Optimal Solution Strategy for Games

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Abstract—**In order to optimize a game, different techniques of optimization are available. Considering the far wide application of game theory it becomes essential to establish the most effective method of optimization of games. In this paper a discussion has been done about the types of games under consideration and the methods to optimize these games. The paper takes into account games of pure and mixed strategies and stable and unstable games. Different techniques of optimization are assessed using various detailed examples and theoretical concepts. By studying the payoffs and cooperation between different players in different techniques the paper establishes a common ground of comparison for testing the efficiency of the techniques .The paper also suggests the advantages and disadvantages of each optimization technique with respect to an algorithmic point of view.**

Keywords-component; Game Theory; Optimization; Nash Equilibrium; Graphical optimization; Linear Programming

I. INTRODUCTION

A game is a competition situation among N persons or groups, called players that is conducted under a prescribed set of rules with known payoffs. The rules define the elementary activities, or moves, of the game. Different players may be allowed different moves, but each player knows the moves available to the other players. Game theory itself is the formal study of the conflict and cooperation. Game theory concepts are applicable to any actions or processes which contain several agents which are interdependent to attaining similar goals. These agents may be individuals, players, organizations etc. Applying the concepts of game theory a game is a formal model of an interactive situation typically involving players. There are various kinds of games. A broad classification of the games can be into Passive and Dynamic. The passive games deal with a finite set of strategies which can be simultaneously applied by players. The Dynamic or Stochastic games are repeated games with probabilistic transitions. The main difference between the two types of games is the fact that in static games the players have a finite set of pure strategies while in a dynamic game the strategies are infinite. Strategies of a player in a game can be divided into two types i.e. Pure Strategies and Mixed Strategies. In case of a pure strategy there is a predetermined plan that prescribes for a player the sequence of moves and countermoves the player would make during a complete game. A mixed strategy on the other hand is defined by a probability distribution over the set of pure strategy. Games can be described formally at various levels of detail. Game theory effectively describes the conditions of the game and the strategy of the players but it doesn't provide the method for the optimal solution of the game. For example in a game of chess the outcomes can be a white win or a black win or a draw. The game theory describes how these outcomes can be achieved but doesn't outline the process of playing through which to achieve these results. Another example could be of the political parties negotiating over forming a majority. Game theory describes which alliance can form a majority but doesn't describe the negotiation process itself. Cooperative game theory investigates such coalition games with respect to the amount of power held by players. In case of a non cooperative game the game theory explicitly models the process of players making choices of their own interest. Nash equilibrium is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing his or her strategy while the other player keeps his strategy unchanged, then the current set of strategy choices and the corresponding payoffs constitute Nash equilibrium. In case of finite games which are the target of this paper according to Mas-Colell etal "Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived by backward induction. Moreover if no player has the same payoffs at any two terminal nodes then there is a unique Nash equilibrium that can be derived in this manner".[1]

II. DISCUSSION

A. *Pure Strategy*

A pure strategy is a predetermined plan that prescribes for a player the sequence of moves and countermoves he will make during a complete game. An optimal solution to the game is said to be reached if neither player finds it beneficial to alter his strategy. In this case the game is said to be in a state of equilibrium. This equilibrium is called as Nash Equilibrium. The game matrix is usually expressed in terms of a payoff to a player. Thus a payoff matrix gives a complete characterization of a game.

For solving two-person zero-sum games we find the Row Minimum (Maximin) and Column Maximum (Minimax) values. These values provide us with the value of the game and determine of a saddle point is present in the game or not. In case the saddle point is found in a game the game is bound to have been played by a pure strategy which would minimize the losses of each player and maximize their profits correlating each other. This saddle point also represents the Nash Equilibrium that will be present in the game. The Nash equilibrium in any game can be achieved by elimination of the unfeasible moves for each player.

	PLAYER I			
		A	B	C
PLAYER J	X	1,0	1,3	3,0
	Y	0,2	0,1	3,0
	Z	0,2	2,4	5,3

Applying elimination of Rows It is clearly noticeable that for Player I strategy C is not feasible compared to A or B and so it is logical to assume that Player I wouldn't play this strategy. This makes it safe to eliminate this strategy. Now for Player J, strategy X and Z clearly dominate strategy Y and hence Y can also be eliminated.

	PLAYER I		
		A	B
	X	1,0	1,3
PLAYER J	Z	0,2	2,4

Now considering this new matrix, for Player I the optimal strategy is B and so Column A can also be eliminated. Similarly Row X can also be eliminated for Player J. This shows that for Player I the optimal strategy is B while for Player J it is Z. This solution to any game is always a stable one and called as the Nash Equilibrium. Considering another 3X3 payoff matrix we calculate the Maximin and the Minimax so as to find the saddle point of the game if any.

	PLAYER A			
PLAYER B	5	3	1	**1**
	7	4	5	**4(Maximin)**
	1	2	6	**1**
	7	**4(Minimax)**	**6**	

In the payoff matrix if Player A chooses his first strategy he can lose at the maximum of 7[5, 7, 1]. For Player B, playing the first strategy he can guarantee a minimum gain of 1[5, 3, 1]. Now if Player A plays his second strategy he can lose a maximum of 4[3, 4, 2] and a maximum of 6[1, 5, 6] if he plays his third strategy. Player B playing his second and third strategy can gain a minimum of 4[7, 4, 5] and 1 [1, 2, 6]. Thus the maximum value in each column represents the maximum loss that Player A will have to accommodate playing that strategy. The minimum value in each row represents the minimum gain that Player B can get playing that strategy. Player A by selecting the second strategy is minimizing his loss while Player B by selecting his second strategy is maximizing his gain. In the above case the Maximin (M_1) is equal to the Minimax (M_2).This equality gives us the saddle point of the game. The condition of optimality is reached here since both players are not tempted to change their strategies. This saddle point is also the value of the game. The value of the game satisfies the following inequality.

$$M_1 \leq \text{Value Of Game} \leq M_2$$

If this inequality doesn't hold the game is a said to be a stable game otherwise it's called an unstable game. Every stable game has a unique value and an optimal (pure) strategy for either player. The optimal strategies can be alternative and not unique. [2,3]

B. Mixed Strategies

In case when the game is unstable and no saddle point is reached, there is no pure strategy for a player. To solve these unstable games, mixed strategies are used. According to these since there is no pure strategy for a player in an unstable game, the player will play a multiple number of strategies to achieve optimality. This condition of optimality can be found out provided that the random payoff is replaced by its expected value. For a game with two players A and B, the mixed strategy for A would be vector A.

$A= [a_1, a_2, a_3 ...a_n]^T$ where a_n is the probability of selecting the n^{th} strategy.

$$\sum_{i=1}^{n} ai = 1$$

Similarly for player B the mixed strategy would be vector B.

$B= [b_1, b_2, b_3 ...b_n]^T$ where b_n is the probability of selecting the n^{th} strategy.

$$\sum_{j=1}^{n} bj = 1$$

If we represent the i,j th entry of the payoff matrix of the game as xij, the payoff matrix would look like the following –

	PLAYER B			
PLAYER A	x11	x12	...	x1j
	x21	x22	...	x2j
	...	...	...	...
	xi1	xi2	...	xij

Considering the Minimax- Criterion for a mixed strategy i.e. the maximum value of the minimum expected gain for Player

A and the minimum value of the maximum expected loss for player B.

For any game matrix there exist optimal strategies X and Y such that

E(X,Y)=A=B=G

Where E(X, Y) = $\sum_{i=1}^{m}\sum_{j=1}^{n} gij\,xij\,yij$, aA and B are the vectors representing the probability distribution of the strategies of the players and G is the expected value of the game. Accordingly every pure strategy game is a special case of the mixed strategy game. Considering that almost all practical games and their application contain mixed strategy, analyzing their optimal solution is of critical importance to the solution of games. [2,3]

III. GRAPHICAL SOLUTION OF TWO PERSON ZERO-SUM GAMES

Graphical solution of a game provides an easy and efficient method of optimizing the game. The pros of Graphical methods are easy approach and simple method of solving while the cons are that they are only applicable to games where at least one of the players has only two strategies. Also they are not considerate from an algorithmic point of view.

Considering the following 2x2 game, assuming there is no saddle point in the game.

	PLAYER B				
PLAYER A		y1	y2	...	yn
	x1	a11	a12	...	a1n
	x2	a21	a22	...	a2n

Since A has two strategies, x1+x2=1 or x2=x1-1 where x1,x2≥0. The expected payoffs to the pure strategy of B is (a1n-a2n)x1+a2n where n is the nth pure strategy. Accordingly the game can be optimized by plotting the graph for x1.

Example

Considering the following 2x3 game

	PLAYER B			
PLAYER A		j=1	j=2	j=3
	i=1	4	1	3
	i=2	2	3	4

The above game doesn't have a saddle point. Therefore the expected payoff of Player A corresponding to the pure strategies of B are given by

TABLE I. EXPECTED PAYOFF TABLE

Strategy Of B	Expected Payoff of A
1	(4-2)x1+2
2	(1-3)x1+3
3	(3-4)x1+4

Graphical representation of the games may be viewed as a Tournament game played on a directed network. In this setting, there are two kinds of nodes: terminating and continuing. Terminating nodes lead to no other node, and player I receives the payoff associated with his/her own arcs. Continuing nodes lead to at least one additional node. The two players simultaneously choose one node each. For the example, the corresponding graph is given in the figure below. In this directed network the edge goes from vertex u to vertex w, and if one player chooses w while the other chooses u, the player who selects w, which is at the head of arc connecting the two nodes, receives payoff $t_{uw.}$

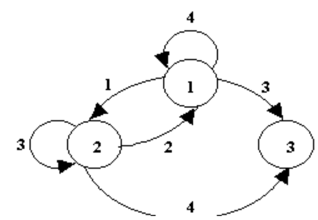

Figure 1. *Graphical Representation Of Two Person Zero-Sum Game*

Plotting the lines as a function of x1. The Maximin occurs at x1=1/4 viz. the point of intersection of the lines of the 1st and 2nd strategy. Therefore A's optimal strategy is x1=1/2, x2=1-x1=> ¾.

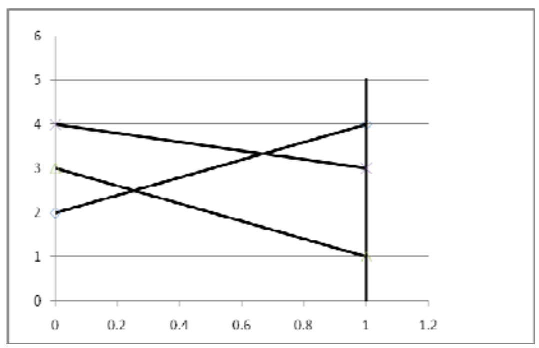

Figure 2. *Graph of x1for maximin*

The value of the game can be determined by keeping the value of x1 in the 1st and 2nd equations.

2(1/2)+2=5/2

-2(1/2)+3=5/2

The optimal strategy of B is y1=1/2,y2=1/2,y3=0.

i=1, 2 and j=1, 2 are essential strategies for Player A and B while j=3 is non essential for Player B.[4,5]

IV. LINEAR PROGRAMMING OPTIMIZATION

In Linear programming problems the objective is either maximization or minimization of the objective function. This optimization is governed by the constraints and the non negativity conditions. This is precisely the condition in games. Each player is trying to maximize his profit and minimize his loss. Accordingly every finite two person zero sum game can be converted into a linear programming problem. With each linear programming problem there is associated a dual of the problem. The optimal values of the objective functions of the two linear problems are equal, corresponding to the value of the game. When solving LP by simplex-type methods, the optimal solution of the dual problem also appears as part of the final tableau. When both players select their optimal strategies one player's highest expected gain is the other player's highest expected loss or the dual of the first players programming problem. Suppose that player B is permitted to adopt mixed strategies, but player A is allowed to use only pure strategies. What mixed strategies Y = (y1, y2, y3) should player B adopt to minimize the maximum expected payoff v? A moment's thought shows that player B must solve the following problem:

$$\text{Min } v = y$$

$$\text{subject to: } \quad T.Y \leq y, \quad U^tY = 1$$

This minimization is over all elements of the decision vector Y ≥ 0, the scalar y is unrestricted in sign, and U is an n-dimensional column vector with all elements equal to one. The left hand side of the first n constraints, by definition, is player B's expected return against player A's pure strategies. It turns out that these mixed strategies are still optimal if we allow player I to employ mixed strategies. Hence by solving for any one of the player's objective the other player's objective can be automatically found.

Player A's objective strategy can be represented as

$$\max\left[A = \min\left\{\sum_{i=1}^{m} ai1xi , \sum_{i=1}^{m} a12xi, \ldots, \sum_{i=1}^{m} ainxi\right\}\right]$$

Subject to the constraints x_i>=0, i=1, 2...m and $\sum_{i=1}^{m} xi$:

Then for Player A the problem becomes of Maximization of Z subject to the constraints.

Max Z= $x_1+x_2+....+x_m$

subject to

$a_{11}x_1+a_{21}x_2+...+a_{m1}x_m \geq 1$

$a_{21}x_1+a_{22}x_2+...+a_{m2}x_m \geq 1$

.

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$a_{1n}x_1+a_{2n}x_2+...+a_{mn}x_m \geq 1$

all $x_i \geq 0$ for i=1,2,...m where Z=1/A

While Player B's objective strategy can be represented as

$$\min\left[B = \max\left\{\sum_{j=1}^{n} a1jyj , \sum_{j=1}^{n} a1jyj, \ldots, \sum_{j=1}^{n} amjyj\right\}\right]$$

Subject to the constraints $y_j \geq 0$, j=1, 2...n and $\sum_{j=1}^{n} yj$:

Then for Player B the problem becomes of Minimization of W subject to the constraints.

Min W= $y_1+y_2+....+y_n$

Subject to

$a_{11}y_1+a_{12}y_2+...+a_{1n}y_n \leq 1$

$a_{21}y_1+a_{22}y_2+...+a_{2n}y_n \leq 1$

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$a_{m1}y_1+a_{m2}y_2+...+a_{mn}y_m \leq 1$

all $y_i \geq 0$ for i=1, 2,...n where W=1/V

The LPP for B is the dual of A and hence solving any one of them yields the solution of the other. Consider the following 3X3 game which can be represented by the following LPP

	PLAYER B		
PLAYER A	8	3	-1
	-2	-3	7
	-6	4	2

Maximize W = $y_1 + y_2 + y_3$ subject to

$8y_1 + 3y_2 - y_3 \leq 1$

$-2y_1 - 3y_2 + 7y_3 \leq 1$

$-6y_1 + 4y_2 + 2y_3 \leq 1$

where x,y,z≥0

The optimal strategy for B would be W=101/180, y_1=13/180, y_2=41/180, y_3=47/180

V=1/W, $Y_1=y_1/W$, $Y_2=y_2/w$ $Y_3=y_3/W$

V=180/101, Y_1=13/101, Y_2=41/101, Y_3=47/101 [$Y_1+Y_2+Y_3$=1]

The optimal strategy for A is the dual of the above given by

Z=W=101/180, x_1=49/180, x_2=5/36, x_3=3/20

V=1/Z, $X_1=x_1/Z$, $X_2=x_2/Z$, $X_3=x_3/Z$

V=190/101, X_1=49/101, X_2=25/101, X_3=27/101 [$X_1+X_2+X_3$=1].[2,4,6]

V. CONCLUSION

In this paper I studied the different types of games and their technique of optimization using examples and theoretical concepts. Using Graphical and Linear Programming Methods to optimize two person zero-sum games one of the conclusion that can be drawn is that since graphical methods are applicable to only games where at least one of the players has only two strategies. This hinders the use of graphical methods in applications and real life problems since hardly any game involves players with limited strategy. On the other hand every game can be represented as a Linear Program and hence can be optimized as a linear programming problem of maximization or minimization. The presence of the dual of a Linear Programming Problem facilitates the use of linear programming since using one objective function and solving for it the objective of the other player can also be found. From an algorithmic point of view linear programming is more feasible compared to graphical methods since it is modular in nature and works step by step.

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