Bicomplex Version of Laplace Transform

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***Abstract* – In this paper we have studied the bicomplex version of Laplace Transformation (LT), condition of existence and examined the Region of Convergence (ROC) of bicomplex Laplace Transformation geometrically with the help of projection on hyperbolic plane. Also we are promoting some useful properties of Laplace Transform in the bicomplex variable from complex variable. These results can be highly applicable in the field of Signal Processing.**

***Index Terms* – Bicomplex Number, Laplace Transform for complex variable, ROC.**

INTRODUCTION

In 1892, in search for special algebras, Corrado Segre **[1]** (1860-1924) published a paper in which he treated an infinite family of algebras whose elements are commutative generalization of complex numbers called bicomplex numbers, tricomplex numbers, …….etc. Introduction part contains three following Sections. Main work will start from Section 4.

1. Certain Basics Theory of Bicomplex Numbers:

Segre defined a bicomplex number as- $\xi = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3$, where x_0, x_1, x_2, x_3 are real numbers, $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. The set of bicomplex numbers is denoted as C_2. In the theory of bicomplex numbers, the sets of real numbers and complex numbers are denoted as C_0 and C_1 respectively. Thus $C_2 = \{\xi : \xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \ , a_0, a_1, a_2, a_3 \in C_0\}$ Or $C_2 = \{\xi : \xi = z_1 + i_2 z_2 \quad z_1, z_2 \in C_1\}$.

1.1 Idempotent Representation:

There are two non-trivial idempotent elements in C_2, denoted by e_1 and e_2 and defined as- $e_1 = \frac{1 + i_1 i_2}{2}$, $e_2 = \frac{1 - i_1 i_2}{2}$; $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$. Every element of C_2 can be uniquely expressed as a complex combination of e_1 and e_2, viz. $\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$. This representation of a bicomplex number is known as the **Idempotent Representation** of ξ. Further, the complex coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the **Idempotent Components** of the bicomplex number $\xi = z_1 + i_2 z_2$.

1.2 Singular Elements:

An element $\xi = z_1 + i_2 z_2$ is singular if and only if $\left|z_1^2 + z_2^2\right| = 0$. The set of singular elements is denoted as O_2 and is characterized as- $O_2 = \{\xi \in C_2 : \xi \text{ is the collection of all} - \text{complex multiples of } e_1 \text{ and } e_2\}$

1.3 Norm:

The norm $\| \cdot \| : C_2 \to C_0^+$ of a bicomplex number is defined as follows: (where C_0^+ denote the set of all non-negative real numbers). If $\xi = z_1 + i_2 z_2 \in C_2$ then -

$\|\xi\| = \left\{|z_1|^2 + |z_2|^2\right\}^{1/2} = \left[\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}\right]^{1/2} = \left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right]^{1/2}$. C_2 is Banach space which is not Banach algebra because, in general, $\|\xi\eta\| \le \sqrt{2}\|\xi\|\|\eta\|$ holds instead of the standard condition, viz. $\|\xi\eta\| \le \|\xi\|\|\eta\|$ In this sense, $\langle C_2, +, \cdot, \times, \|.\| \rangle$ is treated as a modified Banach algebra.

1.4 Auxiliary Complex Spaces:

The Auxiliary complex spaces A_1 and A_2 are defined as follows: $A_1 = \{z_1 - i_1 z_2 \quad \forall\, z_1, z_2 \in C_1\}, A_2 = \{z_1 + i_1 z_2 \quad \forall\, z_1, z_2 \in C_1\}$

1.5 Cartesian Set:

A cartesian set determined by X_1 and X_2 in A_1 and A_2 respectively is denoted as $X_1 \times_e X_2$ and is defined as:
$X_1 \times_e X_2 = \{ z_1 + i_2 z_2 \in C_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2,\ w_1 \in X_1,\ w_2 \in X_2 \}$.
By the help of idempotent representation we define some functions such as- $h_1 : C_2 \to A_1$, $h_2 : C_2 \to A_2$, as follows:-
$p_1(z_1 + i_2 z_2) = h_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1 \quad \forall \quad z_1 + i_2 z_2 \in C_2;$
$p_2(z_1 + i_2 z_2) = h_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2 \quad \forall \quad z_1 + i_2 z_2 \in C_2;$
Hyperbolic projection: $H_\rho (\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3) = a_0 + i_1 i_2 a_3$

2. Certain Basics of Bicomplex Analysis:

In 1928 and 1932, Michiji Futagawa originated the concept of holomorphic functions of a bicomplex variable, in a series of papers **[3], [4].** In 1934, Dragoni **[5]** gave some basic results in the theory of bicomplex holomorphic functions. A full account of the updated theory can be hald from Price**[6].** Some glimpses of the richness of the theory can be seen in Srivastava **[7].**

2.1 Statement: $F(\xi)$ is convergent in domain D iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ are convergent in domain $P_1 : D \to D_1$ and $P_2 : D \to D_2$ respectively.

3. Some Results of Bicomplex numbers-

1. $e^{\xi} = e^{\xi_1 e_1 + \xi_2 e_2} = e^{\xi_1} e_1 + e^{\xi_2} e_2$

2. $\cos \xi = \cos(\xi_1 e_1 + \xi_2 e_2) = (\cos \xi_1) e_1 + (\cos \xi_2) e_2$

3. $\sin \xi = \sin(\xi_1 e_1 + \xi_2 e_2) = (\sin \xi_1) e_1 + (\sin \xi_2) e_2$

4. $\xi^n = (\xi_1 e_1 + \xi_2 e_2)^n = \xi_1^{\,n} e_1 + \xi_2^{\,n} e_2$

5. $(\xi - \eta)^n = (\xi_1 e_1 + \xi_2 e_2 - \eta_1 e_1 - \eta_2 e_2)^n = \left[(\xi_1 - \eta_1) e_1 + (\xi_2 - \eta_2) e_2 \right]^n = (\xi_1 - \eta_1)^n e_1 + (\xi_2 - \eta_2)^n e_2$

6. $\dfrac{\xi}{\eta}$; $\eta \notin O_2 = \dfrac{\xi_1 e_1 + \xi_2 e_2}{\eta_1 e_1 + \eta_2 e_2} = \dfrac{\xi_1}{\eta_1} e_1 + \dfrac{\xi_2}{\eta_2} e_2$

7. $\xi \times \eta = (\xi_1 e_1 + \xi_2 e_2)(\eta_1 e_1 + \eta_2 e_2) = \xi_1 \eta_1 e_1 + \xi_2 \eta_2 e_2$

8. $\xi^n + \eta^n = \xi_1^{\,n} e_1 + \xi_2^{\,n} e_2 + \eta_1^{\,n} e_1 + \eta_2^{\,n} e_2 = (\xi_1^{\,n} + \eta_1^{\,n}) e_1 + (\xi_2^{\,n} + \eta_2^{\,n}) e_2$

9. $\int_D f(\xi)\, d\xi = \int_{D_1} f_{e_1}(\xi_1)\, d\xi_1\ e_1 + \int_{D_2} f_{e_2}(\xi_2)\, d\xi_2\ e_2$; Here $P_1 : D \to D_1$, $P_2 : D \to D_2$

10. $\dfrac{d}{d\xi} f(\xi) = \dfrac{d}{d\xi_1} f_{e_1}(\xi_1) e_1 + \dfrac{d}{d\xi_2} f_{e_2}(\xi_2) e_2$

4. Conjuncture:-

Let $f(t)$ be a real valued function which has exponential order K such that $L[f(t)] = \int_0^\infty f(t) e^{-S_1 t} dt = F(S_1)$ exist here $S_1 \in C_1$ and convergent for $R_e(S_1) > K$ and take another Laplace Transform for $S_2 \in C_1$ such as $L[f(t)] = \int_0^\infty f(t) e^{-S_2 t} dt = F(S_2)$ exist and convergent for $R_e(S_2) > K$. Now we have a linear combination of $F_1(S_1)$ and $F_2(S_2)$ with e_1 and e_2 such as:-

$F_1(S_1) e_1 + F_2(S_2) e_2 = \int_0^\infty f(t) e^{-S_1 t} dt\ e_1 + \int_0^\infty f(t) e^{-S_2 t} dt\ e_2 = \int_0^\infty f(t) e^{-(S_1 e_1 + S_2 e_2)t} dt = \int_0^\infty f(t) e^{-\xi t} dt = F(\xi)$

$F(\xi)$ exist for $R_e(S_1) > K$ and $R_e(S_2) > K$ or $R_e(P_1 : \xi) > K$ and $R_e(P_2 : \xi) > K$.

Since $F_1(S_1)$ and $F_2(S_2)$ are complex valued functions which are convergent for $R_e(S_1) > K$ and $R_e(S_2) > K$ respectively, so a bicomplex valued function $F(\xi) = F_1(S_1) e_1 + F_2(S_2) e_2$ will be convergent in the region D and this is define as: $D = \{ \xi : \xi = S_1 e_1 + S_2 e_2 \ ; R_e(P_1 : \xi) > K \ \&\ R_e(P_2 : \xi) > K \}$.

Let $S_1 = x_1 + i_1 x_2$ & $S_2 = x_3 + i_1 x_4$. Thus $R_e(S_1) = x_1 > K$ & $R_e(S_2) = x_3 > K$, then

$\xi = (x_1 + i_1 x_2) e_1 + (x_3 + i_1 x_4) e_2 = (x_1 + i_1 x_2)\left(\dfrac{1 + i_1 i_2}{2}\right) + (x_3 + i_1 x_4)\left(\dfrac{1 - i_1 i_2}{2}\right) = \dfrac{x_1 + x_3}{2} + \left(\dfrac{x_2 + x_4}{2}\right) i_1 + \left(\dfrac{x_4 - x_2}{2}\right) i_2 + \left(\dfrac{x_1 - x_3}{2}\right) i_1 i_2$

Now there are three possible cases:

1. If $x_1 = x_3$ then $\frac{x_1 - x_3}{2} = 0$ and $\frac{x_1 + x_3}{2} = x_1 = x_3 > K$

Hence if $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2$, then $a_0 > K$ and $a_3 = 0$. Thus $R_e(P_1 : S) = R_e(P_2 : S) = a_0$.

2. If $x_1 > x_3$ then $\frac{x_1 - x_3}{2} > 0 \Rightarrow x_1 > x_3$ and

$\frac{x_1 + x_3}{2} > \frac{K + x_1}{2} > \frac{K + x_1}{2} + \frac{K - x_3}{2} = K + \frac{x_1 - x_3}{2} > K > K - \frac{x_1 - x_3}{2}$ thus $a_0 > K + a_3 > K > K - a_3$ and $a_3 > 0$.

3. If $x_1 < x_3$ then $\frac{x_1 - x_3}{2} < 0 \Rightarrow x_1 < x_3$ and

$\frac{x_1 + x_3}{2} > \frac{K + x_3}{2} > \frac{K + x_3}{2} + \frac{K - x_1}{2} = K - \frac{x_1 - x_3}{2} > K > K + \frac{x_1 - x_3}{2}$ thus $a_0 > K - a_3 > K > K + a_3$ and $a_3 < 0$.

These three conditions making three sets.

$$D_1 = \{\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \ : a_0 > K \ \& \ a_3 = 0\}$$
$$D_2 = \{\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \ : a_0 > K + a_3 \ \& \ a_3 > 0\}$$
$$D_3 = \{\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \ : a_0 > K - a_3 \ \& \ a_3 < 0\}$$

Thus $R_e(P_1 : \xi) > K$ and $R_e(P_2 : \xi) > K$ implies $\xi \in D = D_1 \cup D_2 \cup D_3$ conditions in the set D_1, D_2 and D_3 can be written as $a_0 > K + |a_3|$ and this shows that $\xi \in D$ and D defined as –

$D = \{\xi : H_\rho(\xi)$ represent a Right half plane $a_0 > K + |a_3|\}$

$H_\rho(\xi) \rightarrow$ Hyperbolic projection of ξ.

If $K > -|a_3|$ then hyperbolic projection of ξ lie in the region as given in below figure 1., If $K < -|a_3|$ then hyperbolic projection of ξ lie in the figure 2.

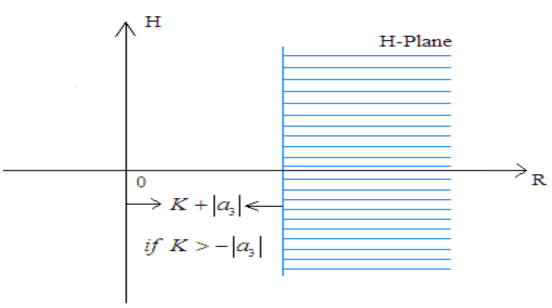

Fig. 1.

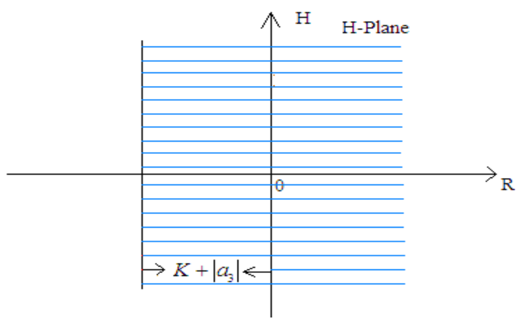

Fig. 2.

If $a_0 = K + |a_3|$ then $F(\xi)$ has poles , these poles are shown on the red lines as in following figures.

If $K > 0$:-

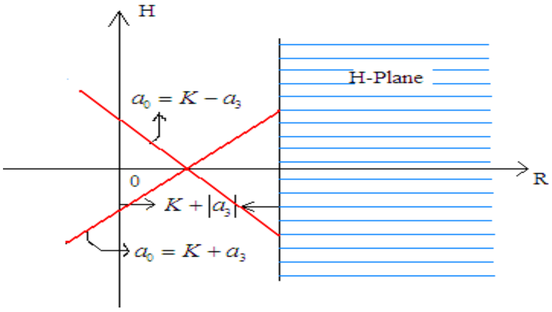

Fig.3.

If $K = 0$

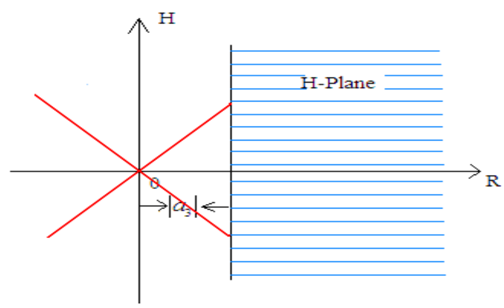

Fig. .4.

If $K < -|a_3|$

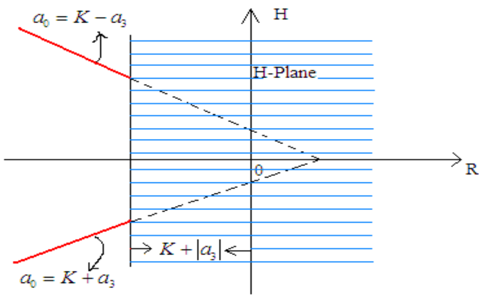

Fig.5.

Converse Result:- It will be the guarantee of existence of Laplace Transform- If $S \in D = D_1 \cup D_2 \cup D_3$ then $R_e(P_1(\xi))$ and $R_e(P_2(\xi))$ are greater than K. First of all we express ξ in independent component form such as-

$\xi = a_0 + i_1 a_1 + i_2 a_2 + j a_3 = (a_0 + i_1 a_1) + i_2(a_2 + i_1 a_3) = [(a_0 + a_3) + i_1(a_1 - a_2)]e_1 + [(a_0 - a_3) + i_1(a_1 + a_2)]e_2$

; $R_e(P_1(\xi)) = a_0 + a_3$ & $R_e(P_2(\xi)) = a_0 - a_3$

Case 1. Let $S \in D_1$ then $a_0 > K$, $a_3 = 0$, $R_e(P_1(\xi)) = a_0 + a_3 = a_0 > K$ .Similarly $R_e(P_2(\xi)) = a_0 > K$.

Case 2. Let $S \in D_2$ then $a_0 > K + a_3$ & $a_3 > 0$ then $a_0 + a_3 > a_0 - a_3 > K$. Hence the result.

Case 3. Let $S \in D_3$ then $a_0 > K - a_3$ & $a_3 < 0$ then $a_0 - a_3 > a_0 + a_3 > K$. Hence the result.

LAPLACE TRANSFORM:-

Now we are ready to define the Laplace Transform for Bicomplex variable. Let $f(t)$ be a real valued function of exponential order K. Then Laplace Transform of $f(t)$ for $t \geq 0$ can be define as- $L\{f(t)\} = \int_0^\infty f(t)e^{-\xi t}dt = F(\xi)$. Here $F(\xi)$ is exist and convergent for all $\xi \in D = D_1 \cup D_2 \cup D_3$ or which has $H_\rho(\xi)$ in the Right half plane $a_0 > K + |a_3|$. In D there are infinite ξ which have same H_ρ hyperbolic projection because a_1 & a_2 are free from restriction.

Theorem 1: (EXISTANCE OF LAPLACE TRANSFORM):-

If $f(t)$ is of exponential order K, then its Laplace transform $L\{f(t)\} = F(\xi)$ is given by- $F(\xi) = \int_0^\infty f(t)e^{-\xi t}dt$, Where $\xi = \xi_1 e_1 + \xi_2 e_2$, the defining integral for $F(\xi)$ exist at points ξ in the Cartesian region of right half planes $R_e(\xi_1) > K$ and $R_e(\xi_2) > K$

Proof:- $F(\xi) = \int_0^\infty f(t)e^{-\xi t}dt = \int_0^\infty f(t)e^{-\xi_1 t}dt\ e_1 + \int_0^\infty f(t)e^{-\xi_2 t}dt\ e_2$ .Both the integrals are exist when $R_e(\xi_1) > K$ and $R_e(\xi_2) > K$. So $F(\xi)$ exist for $\xi = \xi_1 e_1 + \xi_2 e_2$ or $\xi = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3$, where $x_0 - |x_3| > K$.

Theorem 2: (UNIQUENESS OF LAPLACE TRANSFORM):-

Let the function $f(t)$ and $g(t)$ have Laplace Transform $F(\xi)$ and $G(\xi)$, respectively. If $F(\xi) = G(\xi)$, then $f(t) = g(t)$

Proof:- Let- $F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ & $G(\xi) = G_{e_1}(\xi_1)e_1 + G_{e_2}(\xi_2)e_2$. If $F(\xi) = G(\xi)$ then it is possible iff $F_{e_1}(\xi_1) = G_{e_1}(\xi_1)$ & $F_{e_2}(\xi_2) = G_{e_2}(\xi_2) \Rightarrow \int_0^\infty e^{\xi_1 t} f(t)dt = \int_0^\infty e^{\xi_1 t} g(t)dt$

& $\int_0^\infty e^{\xi_2 t} f(t)dt = \int_0^\infty e^{\xi_2 t} g(t)dt$. It's only possible if $f(t) = g(t)$.

PROPERTIES OF LAPLACE TRANSFORM:-

Theorem 3: (***LINEARITY OF LAPLACE TRANSFORM):-***

Let the function $f(t)$ and $g(t)$ have Laplace Transform $F(\xi)$ and $G(\xi)$, respectively. If a and b are constants, then $L\{a\ f(t) + b\ g(t)\} = aF(\xi) + bF(\xi)$

Proof:- Let K be chosen so that both F and G are defined for $\xi \in D = D_1 \cup D_2 \cup D_3$, then

$$L\{a\ f(t)+b\ g(t)\} = \int_0^\infty \{a\ f(t)+b\ g(t)\} e^{-\xi t} dt = \int_0^\infty \{a\ f(t)+b\ g(t)\} e^{-\xi_1 t} dt\ e_1 + \int_0^\infty \{a\ f(t)+b\ g(t)\} e^{-\xi_2 t} dt\ e_2$$

$$= a\int_0^\infty f(t) e^{-\xi_1 t} dt\ e_1 + b\int_0^\infty f(t) e^{-\xi_1 t} dt\ e_1 + a\int_0^\infty f(t) e^{-\xi_2 t} dt\ e_2 + b\int_0^\infty f(t) e^{-\xi_2 t} dt\ e_2$$

$$= aF_{e_1}(\xi_1)e_1 + bG_{e_1}(\xi_1)e_1 + aF_{e_2}(\xi_2)e_2 + bG_{e_2}(\xi_2)e_2 = a\left[F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2\right] + b\left[G_{e_1}(\xi_1)e_1 + G_{e_2}(\xi_2)e_2\right]$$

$$= aF(\xi_1 e_1 + \xi_2 e_2) + bG(\xi_1 e_1 + \xi_2 e_2) = aF(\xi) + bG(\xi)$$

$$Thus\ \boxed{L\{a\ f(t)+b\ g(t)\} = aF(\xi) + bF(\xi)}$$

NOTE:- $F(\xi) = F(\xi_1 e_1 + \xi_2 e_2) = \int_0^\infty f(t) e^{-\xi t} dt = \int_0^\infty f(t) e^{-\xi_1 t} dt\ e_1 + \int_0^\infty f(t) e^{-\xi_2 t} dt\ e_2 = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$

Theorem 4: (***LAPLACE TRANSFORM OF DERIVATIVES):-***

Let $f(t)$ and $f'(t)$ be continuous for $t \ge 0$ of exponential order K, then $L\{f'(t)\} = \xi F(\xi) - f(0)$ where $F(\xi) = L\{f(t)\}$

Proof:- Let K be large enough that both $f(t)$ and $f'(t)$ are of exponential order K. $L\{f'(t)\}$ is given by

$$L\{f'(t)\} = \int_0^\infty f'(t) e^{-\xi t} dt\ ;\ \xi \in D = \left[e^{-\xi t} f(t)\right]_0^\infty - \int_0^\infty -\xi e^{-\xi t} f(t)\ dt$$

$$= -f(0) + \xi\int_0^\infty e^{-\xi t} f(t)\ dt = -f(0) + \xi F(\xi) \Rightarrow\ Thus\ \boxed{L\{f'(t)\} = \xi F(\xi) - f(0)}$$

Corollary:- If $f(t)$, $f'(t)$ and $f''(t)$ are of exponential order, then –

$$L\{f''(t)\} = \int_0^\infty f''(t) e^{-\xi t} dt\ ;\ \xi \in D = \left[e^{-\xi t} f'(t)\right]_0^\infty - \int_0^\infty -\xi e^{-\xi t} f'(t)\ dt = -f'(0) + \xi\int_0^\infty e^{-\xi t} f'(t)\ dt = -f'(0) + \xi\left[L\{f'(t)\}\right]$$

$$= -f'(0) + \xi\left[\xi F(\xi) - f(0)\right] = \xi^2 F(\xi) - \xi f(0) - f'(0)$$

Similarly- $L\{f'''(t)\} = \xi^3 F(\xi) - \xi^2 f(0) - \xi f'(0) - f''(0)$ etc.

Theorem5: (LAPLACE TRANSFORM OF INTEGRATIVES):-

Let $f(t)$ be a continuous for $t \ge 0$ and be of exponential order K and let $F(\xi)$ be its Laplace transform. Then – $L\left\{\int_0^t f(t)dt\right\} = \dfrac{F(\xi)}{\xi}$

Proof:- Let $g(t) = \int_0^t f(t)dt$ then $g'(t) = f(t)$ and $g(0) = \int_0^t f(t)dt = 0$, Since $f(t)$ is of exponential order K, there exist $M > 0\ \&\ K$ so that- $|f(t)| \le Me^{Kt}$.

$|g(t)| \le \int_0^t |f(t)|dt \le M\int_0^t e^{Kt} dt = \dfrac{M}{K}\left(e^{Kt} - 1\right) \le M_1 e^{Kt}$. Thus $g(t)$ is also of exponential order K.

Then- $F(\xi) = L\{f(t)\} = L\{g'(t)\} = \xi L\{g(t)\} - g(o) = \xi L\{g(t)\}$

$$Thus\ \boxed{L\{g(t)\} = \frac{F(\xi)}{\xi} = L\left\{\int_0^\infty f(t)dt\right\}}$$

Theorem 6: (MULTIPLICATION BY t) :–

Prove that $L\{tf(t)\} = -F'(\xi)$

Proof:- We have Leibniz's rule in complex analysis **[8]** Now-

$$F'(\xi) = f_{e_1}'(\xi_1)e_1 + f_{e_2}'(\xi_2)e_2 = \frac{\partial}{\partial\xi_1}\int_0^\infty f(t) e^{-\xi_1 t} dt\ e_1 + \frac{\partial}{\partial\xi_2}\int_0^\infty f(t) e^{-\xi_2 t} dt\ e_2$$

By Leibniz's rule we can write it as-

$$= \int_0^\infty \frac{\partial}{\partial\xi_1} f(t) e^{-\xi_1 t} dt\ e_1 + \int_0^\infty \frac{\partial}{\partial\xi_2} f(t) e^{-\xi_2 t} dt\ e_2 = \int_0^\infty -tf(t) e^{-\xi_1 t} dt\ e_1 + \int_0^\infty tf(t) e^{-\xi_2 t} dt\ e_2 = -\int_0^\infty tf(t) e^{-\xi t} dt = -L(tf(t))$$

$$Thus\ \boxed{L\{tf(t)\} = -F'(\xi)}$$

Theorem 7: (DIVISION BY t) –

let $F(\xi)$ denote the Laplace transform of $f(t)$. If $\lim_{t\to 0}\frac{f(t)}{t}$ exist, then - $L\left\{\frac{f(t)}{t}\right\} = \int_{\xi}^{\infty} F(\eta)\,d\eta$

Proof:- We have $F(\eta) = \int_0^{\infty} f(t)e^{-\eta t}dt$.

So $F_{e_1}(\eta_1)e_1 + F_{e_2}(\eta_2)e_2 = \int_0^{\infty} f(t)e^{-\eta_1 t}dt\ e_1 + \int_0^{\infty} f(t)e^{-\eta_2 t}dt\ e_2$.Now we take integration with limit $\xi = \xi_1 e_1 + \xi_2 e_2$ to ∞,

$$\int_{\xi}^{\infty} F(\eta)\,d\eta = \int_{\xi_1}^{\infty} F_{e_1}(\eta_1)\,d\eta_1\ e_1 + \int_{\xi_1}^{\infty} F_{e_2}(\eta_2)\,d\eta_2\ e_1$$

$$= \int_{\xi_1}^{\infty}\int_0^{\infty} f(t)e^{-\eta_1 t}dt\ d\eta_1\ e_1 + \int_{\xi_2}^{\infty}\int_0^{\infty} f(t)e^{-\eta_2 t}dt\ d\eta_2\ e_2$$

We reverse the order of integration in the double integral of the equation to obtain-

$$\int_{\xi}^{\infty} F(\eta)\,d\eta = \int_{\xi_1}^{\infty}\int_0^{\infty} f(t)e^{-\eta_1 t}dt\ d\eta_1\ e_1 + \int_{\xi_2}^{\infty}\int_0^{\infty} f(t)e^{-\eta_2 t}dt\ d\eta_2\ e_2 = \int_0^{\infty}\left[\frac{f(t)}{-t}e^{-\eta_1 t}\right]_{\xi_1}^{\infty} dt\ e_1 + \int_0^{\infty}\left[\frac{f(t)}{-t}e^{-\eta_2 t}\right]_{\xi_2}^{\infty} dt\ e_2$$

$$= \int_0^{\infty}\frac{f(t)}{-t}\left(0 - e^{-\xi_1 t}\right) dt\ e_1 + \int_0^{\infty}\frac{f(t)}{-t}\left(0 - e^{-\xi_2 t}\right) dt\ e_2 = \int_0^{\infty}\frac{f(t)}{t}e^{-\xi_1 t}\ dt\ e_1 + \int_0^{\infty}\frac{f(t)}{t}e^{-\xi_2 t}\ dt\ e_2$$

$$= \int_0^{\infty}\frac{f(t)}{t}e^{-\xi t}\ dt = L\left\{\frac{f(t)}{t}\right\}$$

$$Thus\quad \boxed{L\left\{\frac{f(t)}{t}\right\} = \int_{\xi}^{\infty} F(\eta)\,d\eta}$$

Theorem 8: (SHIFTING THE VARIABLE ξ)-

If $F(\xi)$ is the Laplace transform of $f(t)$, then – $L\left\{e^{at} f(t)\right\} = F(\xi - a)$

Proof:- we have $L\{f(t)\} = F(S) = \int_0^{\infty} f(t)e^{-\xi t}dt \Rightarrow L\left\{e^{at} f(t)\right\} = \int_0^{\infty} e^{at}e^{-\xi t} f(t)dt = \int_0^{\infty} e^{-(\xi-a)t} f(t)dt = F(\xi - a)$

$$Thus\quad \boxed{L\left\{e^{at} f(t)\right\} = F(\xi - a)}$$

Theorem 9: (SHIFTING THE VARIABLE t):-

If $F(\xi)$ is the Laplace transform of $f(t)$ and $a \ge 0$, then $L\left\{U_a(t)\, f(t-a)\right\} = e^{-a\xi}F(\xi)$, where $f(t)$ and $U_a(t)\, f(t-a)$ are illustrated in figure-

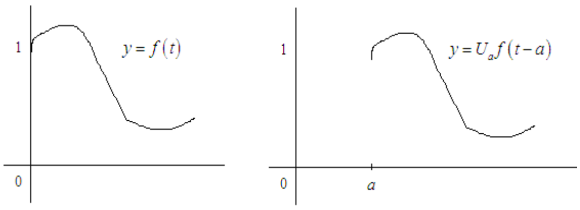

Fig. 6.

Proof:-

We have $e^{-a\xi}F(\xi) = e^{-a\xi}\int_0^{\infty} f(S)e^{-\xi s}ds = \int_0^{\infty} f(S)e^{-(a+s)\xi}ds$

Let $a + s = t$, then- $e^{-a\xi}F(\xi) = \int_a^{\infty} f(t-a)e^{-\xi t}dt = \int_a^{\infty} f(t-a)e^{-\xi_1 t}dt\ e_1 + \int_a^{\infty} f(t-a)e^{-\xi_2 t}dt\ e_2$

Because $U_a(t)\, f(t-a) = 0$ for $t < a$ and $U_a(t)\, f(t-a) = f(t-a)$ for $t \ge a$, then

$$e^{-a\xi}F(\xi) = \int_0^{\infty} U_a(t)\, f(t-a)e^{-\xi_1 t}dt\ e_1 + \int_0^{\infty} U_a(t)\, f(t-a)e^{-\xi_2 t}dt\ e_2 = \int_0^{\infty} U_a(t)\, f(t-a)e^{-\xi t}dt$$

$$Thus\quad \boxed{L\left\{U_a(t)\, f(t-a)\right\} = e^{-a\xi}F(\xi)}$$

NOTE:-

$$\int_0^{\infty} U_a(t)\, f(t-a) e^{-\xi_1 t} dt = \int_0^{a} U_a(t)\, f(t-a) e^{-\xi_1 t} dt + \int_a^{\infty} U_a(t)\, f(t-a) e^{-\xi_1 t} dt = 0 + \int_a^{\infty} f(t-a) e^{-\xi_1 t} dt$$

Some Laplace Transform Pairs with ROC in Complex and Bicomplex Variable are given in the following table-

- In this table we use- $a = a_0 + i_1 a_1 \in C_1$, $S = S_0 + i_1 S_1 \in C_1$, $\xi = \xi_0 + i_1\xi_1 + i_2\xi_2 + i_1 i_2 \xi_3 = \xi^1 e_1 + \xi^2 e_2 \in C_2$.
- $U(t)$- Unit step Function, $\delta(t)$- Unit Impulse function.

S. No.	$f(t)$	Exponential order	$F(S)$	$F(\xi)$	ROC of $F(S)$	ROC of $F(\xi)$	
						With idempotent Components	With Real Components
1.	$\delta(t)$	0	1	1	All S	All ξ	All ξ
2.	$U(t)$	0	$1/S$	$1/\xi$	$R_e(S) > 0$ Or $S_0 > 0$	$R_e(\xi^1) > 0$ & $R_e(\xi^2) > 0$	$\xi_0 > \lvert\xi_3\rvert$
3.	$-U(-t)$	0	$1/S$	$1/\xi$	$R_e(S) < 0$ Or $S_0 < 0$	$R_e(\xi^1) < 0$ & $R_e(\xi^2) < 0$	$\xi_0 < \lvert\xi_3\rvert$
4.	$t^m U(t)$	0	$\frac{m!}{S^{m+1}}$	$\frac{m!}{\xi^{m+1}}$	$R_e(S) > 0$	$R_e(\xi^1) > 0$ & $R_e(\xi^2) > 0$	$\xi_0 > \lvert\xi_3\rvert$
5.	$e^{-at}U(t)$	$-a_0$	$\frac{1}{S+a}$	$\frac{1}{\xi+a}$	$R_e(S) > R_e(a) = -a_0$	$R_e(\xi^1) > -a_0$ & $R_e(\xi^2) > -a_0$	$\xi_0 > -a_0 + \lvert\xi_3\rvert$
6.	$-e^{-at}U(-t)$	$-a_0$	$\frac{1}{S+a}$	$\frac{1}{\xi+a}$	$R_e(S) < -a_0$	$R_e(\xi^1) < -a_0$ & $R_e(\xi^2) < -a_0$	$\xi_0 < -a_0 + \lvert\xi_3\rvert$
7.	$t^m e^{-at}U(t)$	$-a_0$	$\frac{m!}{(S+a)^{m+1}}$	$\frac{m!}{(\xi+a)^{m+1}}$	$R_e(S) > R_e(a) = -a_0$	$R_e(\xi^1) > -a_0$ & $R_e(\xi^2) > -a_0$	$\xi_0 > -a_0 + \lvert\xi_3\rvert$
8.	$-t^m e^{-at}U(-t)$	$-a_0$	$\frac{m!}{(S+a)^{m+1}}$	$\frac{m!}{(\xi+a)^{m+1}}$	$R_e(S) < -a_0$	$R_e(\xi^1) < -a_0$ & $R_e(\xi^2) < -a_0$	$\xi_0 < -a_0 + \lvert\xi_3\rvert$
9.	$Cos\omega t.U(t)$	0	$\frac{S}{S^2+\omega^2}$	$\frac{\xi}{\xi^2+\omega^2}$	$R_e(S) > 0$	$R_e(\xi^1) > 0$ & $R_e(\xi^2) > 0$	$\xi_0 > \lvert\xi_3\rvert$
10.	$Sin\omega t.U(t)$	0	$\frac{\omega}{S^2+\omega^2}$	$\frac{\omega}{\xi^2+\omega^2}$	$R_e(S) > 0$	$R_e(\xi^1) > 0$ & $R_e(\xi^2) > 0$	$\xi_0 > \lvert\xi_3\rvert$
11.	$e^{-at}Cos\omega t.U(t)$	$-a_0$	$\frac{S+a}{(S+a)^2+\omega^2}$	$\frac{\xi+a}{(\xi+a)^2+\omega^2}$	$R_e(S) > R_e(a) = -a_0$	$R_e(\xi^1) > -a_0$ & $R_e(\xi^2) > -a_0$	$\xi_0 > -a_0 + \lvert\xi_3\rvert$
12.	$e^{-at}Sin\omega t.U(t)$	$-a_0$	$\frac{\omega}{(S+a)^2+\omega^2}$	$\frac{\omega}{(\xi+a)^2+\omega^2}$	$R_e(S) > R_e(a) = -a_0$	$R_e(\xi^1) > -a_0$ & $R_e(\xi^2) > -a_0$	$\xi_0 > -a_0 + \lvert\xi_3\rvert$
13.	$U(t-t_0)$	0	$\frac{e^{-st_0}}{S}$	$\frac{e^{-\xi t_0}}{\xi}$	$R_e(S) > 0$	$R_e(\xi^1) > 0$ & $R_e(\xi^2) > 0$	$\xi_0 > \lvert\xi_3\rvert$
14.	$\delta'(t)$	0	S	ξ	All S	All ξ	All ξ

FUTURE WORK:

We can see the effect on ROC if we take ω as complex and purely imaginary and ROC also affected if we have hyperbolic functions. This paper is promoted for Linear Time Invariant (LTI) systems.

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