ANALYTICAL APPROACH TO DETERMINING OF PARAMETERS WHICH CHARACTERIZE SURFACE LAYER QUALITY OF THE PARTS HARDENED BY A TRAVELING DIAMOND SPHERE

S.R. Abulkhanov, D.L. Skuratov, A.I. Khaimovich, A.N. Shvetcov

Samara State Aerospace University named after academician Korolev S.P. (National Research University) (SSAU), 34 Moscovskoye shosse, Samara, Russia, 443086

Abstract

The present work is dedicated to analysis of a plane boundary problem of a sphere movement along an elastic semi-plane. Moreover it is supposed that in the area of contact between the sphere and the semi-plane there will be sectors with multidirectional frictional load as well as a sector with adhesive bonds. By means of the received solution of the boundary problem we've evaluated the area of plastic deformation below the sphere

based on the condition of equality of load and plasticity invariable (yield point (σ_v)). Plastic deformation

height was considered to be the maximum height of plastic deformations penetration under the traveling sphere. Comparison of experimental and estimated data demonstrated accuracy acceptable for use of the elaborated method in real production.

1. Introduction

Production process of a part surface hardening by its deformation by means of a traveling diamond sphere is distinguished from other surface plastic deformation methods due to resulting stable quality indices of the processed surface.

Basic parameters characterizing the surface layer quality of the parts inclusive of those hardened by a diamond sphere are as follows: surface roughness, residual strain and cold deformation, i.e. distribution of microhardness across the surface layer after its hardening. The methods used nowadays for control of residual strain and distribution of

microhardness (h_H) throughout a part surface layer height result in destruction of the explored surface which is not

acceptable in some cases, in particular during evaluation of large and case parts. That's why development of nondestructive methods for quantitative estimation of the surface layer parameters is a technological problem of current concern.

For the parts being operated under the conditions of friction or repeated loads the height of mechanical hardening

 (h_H) to a large extent is indicative of performance characteristics of the parts [1]. In this respect there was developed a method for analytical determination of the height of a plastically deformed layer of a surface subjected to hardening by means of a diamond sphere.

2. Methodology

Experimental determination of a mechanical hardening profile was performed with use of angle laps at the samples preliminary processed by a diamond sphere. Microhardness was measured on the basis of a reproduced impress formed by indentation of a Vickers diamond point with the force of 2 N in accordance with the standard GOST 9450-76 "Measurement of microhardness by diamond instruments". For measurement of the impress diagonal size the microhardness tester PMT-3 made by OJSC LOMO (Saint-Petersburg), Russia, was used. For determination of the height of the part plastically deformed surface layer the measurements were made approximately at the interval of 10 µm.

The hardened surface layer height was determined as a distance measured along normal to the surface starting from the surface and ending at the place at which geometric dimensions of the Vickers point impress are no longer changed. The rings with the outer diameter of 62 mm and the width of 12 mm made of stainless steel 12X18H9T (the national standard GOST 5632-72 "Superalloyed steels and corrosion-resistant, heat-resistant and heatproof alloys. Makes") were taken as samples.

The samples were hardened by a diamond sphere at a threading lathe. The force of pressing the diamond sphere against the processed surface was ensured by a compression spring with definite preset force.

Five samples were used for investigation of each hardening regimen. After determining the mechanical hardening height its mean value was found out as well as the maximum and minimum height of the plastically deformed layer.

3. Problem statement. Plane elastic problem of introduction of a sphere into an elastic semi-plane

The height and degree of cold deformation at time of diamond burnishing depend on the processing regimens: pressing force of a tool against a workpiece (P_y) , burnishing speed (\mathcal{V}) , radius (imprint) of a diamond sphere (R_{sph}) , as well as on physical and mechanical properties of the processed material: elastic modulus (E), the Poisson ratio (\mathcal{V}) and yield point (σ_y) [1-2]. Due to this analytical determination of the width of a cold-hardened layer in relation to the above mentioned parameters is an issue of undoubted interest.

The diamond sphere radius is much more less then the radius of the hardened surface curvature, that's why we'll admit that the diamond tool travels along the plane (of the part surface).

Two opposed frictional forces F_{FR} as well as an area with adhesive bonds between the tool and the processed surface occur at the contact surface in the course of travel of the diamond tool being introduced into metal under action of normal force (P_v) [1] (Fig. 1).



Figure. 1. Diagram of the surface deformation by the diamond sphere

For mathematical formalization of the set problem let assume the following load distribution diagram of the elastic semi-plane (Fig. 2).



Fig. 2. Mathematically formalized diagram of the surface deformation by the diamond sphere

It worth mentioning that the elastic semi-space N in the problem under consideration will have the line $A_1 A_5 A_4$ with the excluded cavity $A_1 A_4$ as a boundary.

Finding solution of the boundary problem under consideration will be made by the method of functions of complex variable. In this case for conformal mapping of N area onto the lower semi-plane it is necessary to use the following transformation [3, 7-9]:

$$z_{z_1} = z + \frac{\sigma \cdot e^{z\alpha}}{8\pi} \cdot \frac{\left(i + z \cdot e^{-z\alpha}\right)^3}{1 - z \cdot e^{-z\alpha}}.$$
(1)

The above transformation will result in the following load distribution diagram (Fig. 3).



Fig. 3. Load distribution at the area of contact between the diamond sphere and the semi-plane surface

Let's investigate the plane problem for the elastic semi-plane along which the spherical tool impressed against the semi-plane with the preset force is traveling. Let's introduce two functions of complex variable determined in the lower semi-plane and represented by the Cauchy-like integrals the values of density of which are equal correspondingly to the values of normal and tangential forces applied to the semi-plane boundary [4-6]:

$$w_{1}(z) = u_{1} + iv_{1} = \int_{-\infty}^{+\infty} (\sigma_{y})_{y=0} \frac{d\xi}{\xi - z} \left[u_{2}(z) = u_{2} + iv_{2} = \int_{-\infty}^{+\infty} (\tau_{xy})_{y=0} \frac{d\xi}{\xi - z} \right]$$

Let's set the boundary conditions:

$$\begin{bmatrix} a_1 a_5 a_4 \end{bmatrix}: \quad \sigma_y = 0, \ \tau_{xy} = 0 - area \ free \ from \ external \ loads; \\ \begin{bmatrix} a_1 a_2 \end{bmatrix}: \quad v_y = \Phi'_x z'_{z_1}, \ \tau_{xy} - f_1 \cdot \sigma_y = 0 - area \ with \ friction; \\ \begin{bmatrix} a_2 a_3 \end{bmatrix}: \quad u_x(x) = 0, \ v_y = \Phi'_x z'_{z_1} - area \ with \ adhesive \ bonds; \\ \begin{bmatrix} a_3 a_4 \end{bmatrix}: \quad v_y = \Phi'_x z'_{z_1}, \ \tau_{xy} + f_2 \cdot \sigma_y = 0 - area \ with \ friction,$$

where $v_y(x)$, $u_x(x)$ - displacement along the axes y and x correspondingly; τ_{xy} and σ_y - tangential and normal strain; F(x) - function describing the diamond sphere surface; z'_{z_1} - derivative of the conformal mapping function (1). f_i here is a friction ratio.

Taking into account that the length of $[a_1 a_4]$ segment is much more less then the diamond sphere diameter, the diamond sphere surface contacting the semi-plane $N(z_1)$ may be approximated by a parabola fragment, i.e. $F(x) = \frac{x^2}{2\Sigma}$. Where R_{sph} – the diamond sphere radius, $N(z_1)$ – the lower semi-plane.

$$(X) = \frac{1}{2R_{sph}}$$

Use of functions $w_1(z)$ and $w_2(z)$, $(w_j = u_j + iv_j)$, where $j = \overline{1,2}$, [4-7]) allows to define the boundary conditions of the considered contact problem at $[a_1 a_5 a_4]$ segment:

$$\nu_1 = \operatorname{Im} w_1 = 0, \ \nu_2 = \operatorname{Im} w_2 = 0;$$
 (3)

at [*a*₁ *a*₂]:

$$\nu_{2} - f_{1}\nu_{1} = \operatorname{Im}(w_{2} - f_{1}w_{1}) = 0,$$

$$u_{1} + \beta\nu_{2} = \operatorname{Im}(iw_{1} + \beta w_{2}) = \frac{GN - MH}{CN - DM}F'_{x}z'_{z_{1}} = 0;$$
⁽⁴⁾

at [*a*₃ *a*₄]:

$$u_{2} + f_{2}v_{1} = \operatorname{Im}(w_{2} + f_{2}w_{1}) = 0,$$

$$u_{1} + \beta v_{2} = \operatorname{Im}(iw_{1} + \beta w_{2}) = \frac{GN - MH}{CN - DM} F'_{x}z'_{z_{1}} = 0.$$
⁽⁵⁾

Note that the results of work [1] show that $f_1 = f_2 = f$. Different signs of the first equations (4) and (5) are conditioned by different directions of the friction forces at the segments $[a_1 \ a_2]$ and $[a_3 \ a_4]$. The coefficients *C*, *N*, *D*, *M*, *H*, *G* depend on the diamond sphere (tool travel) speed and elastic response of the semiplane being deformed. Specific formulation of these invariables is given in [4-6].

The boundary condition at the segment $[a_2 a_3]$ corresponding to the area with adhesive bonds is formulated as follows:

$$u_{2} - \beta' v_{1} = \operatorname{Im}(iw_{2} - \beta'w_{1}) = 0,$$

$$u_{1} + \beta v_{2} = \operatorname{Im}(iw_{1} + \beta w_{2}) = \frac{GN - MH}{CN - DM} F'_{x} z'_{z_{1}} = 0.$$
⁽⁶⁾

In the expressions (6) $\beta \neq \beta'$, this is caused by the situation when movement of a diamond sphere (tool) gives rise to some "anisotropy" of a deformation pattern. Specific formulation of the coefficient β used in the boundary conditions (4), (5) and (6) is given in [4-7]. Let's determine the coefficient β' which characterizes displacement *u* along the axis *x*.

For a traveling tool in accordance with [4, 5]:

$$u = i \left[A \varphi(z_1) - A \overline{\varphi}(\overline{z}_1) + B \psi(z_2) - B \overline{\psi}(\overline{z}_2) \right]$$
$$\frac{du}{dx} = i \left[A \varphi'(z_1) - A \overline{\varphi'}(\overline{z}_2) + B \psi'(z_2) - B \overline{\psi'}(\overline{z}_2) \right]$$

then

or

$$\frac{du}{dx} = 2 \operatorname{Im}[A \varphi'(z_1) - B \psi'(z_2)].$$

Let's transform the last expression by using the expression for $\varphi'(z_1)$ and $\psi'(z_2)$ from [4, 5]

$$\frac{du}{dx} = 2 \operatorname{Im} \left[\frac{AN + BM}{GN - MH} \int_{-\infty}^{+\infty} \frac{(\sigma_y)_{y=0} d\xi}{\xi - z_1} - \frac{BG + AH}{GN - MH} \int_{-\infty}^{+\infty} \frac{(\tau_{xy})_{y=0} d\xi}{\xi - z_2} \right].$$

Let's direct z_1 and z_2 at y = 0 to a random point at the axis x and by making use of the boundary values characteristic of the Cauchy-like integrals we'll get

$$\frac{du}{dx} = 2 \operatorname{Im} \left[\frac{AN + BM}{GN - MH} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(\sigma_y)_{y=0} d\xi}{\xi - x} - \frac{i}{2} (\sigma_y)_{y=0} \right) - \frac{BG + AH}{GN - MH} \left(\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{(\tau_{xy})_{y=0} d\xi}{\xi - x} + \frac{1}{2} (\tau_{xy})_{y=0} \right) \right]$$

or

$$\frac{du}{dx} = 2 \operatorname{Im}\left[\frac{AN + BM}{GN - MH} \left(\sigma_{y}\right)_{y=0} - \frac{BG + AH}{GN - MH} \left(\int_{-\infty}^{+\infty} \frac{\left(\tau_{xy}\right)_{y=0} d\xi}{\xi - x}\right)\right].$$

Based on which and by use of $W_1(z)$ and $W_2(z)$ we'll get the resulting:

$$\frac{du}{dx} = \frac{AN + BM}{GN - MH}u_2 - \frac{BG + AH}{GN - MH}v_1$$

Since at the segment $[a_2 a_3] u = 0$ the last expression may be formulated as

$$u_2 - \frac{BG + AH}{AN + BM}v_1 = 0$$

i.e. $\beta' = -\frac{BG + AH}{AN + BM}$, where analytical formulation of the coefficients *G*, *H*, *N*, *M*, *A*, *B* is taken from [4-6].

Therefore the boundary conditions of the contact problem under consideration will be formulated as follows:

$$\begin{bmatrix} a_{1}a_{5}a_{4} \end{bmatrix}: \quad \operatorname{Im}w_{1} = 0, \quad \operatorname{Im}w_{2} = 0; \\ \begin{bmatrix} a_{1}a_{2} \end{bmatrix}: \quad \operatorname{Im}(w_{2} - fw_{1}) = 0, \quad \operatorname{Im}(iw_{1} + \beta w_{2}) = \mathrm{F}; \\ \begin{bmatrix} a_{2}a_{3} \end{bmatrix}: \quad \operatorname{Im}(iw_{2} - \beta'w_{1}) = 0, \quad \operatorname{Im}(iw_{1} + \beta w_{2}) = \mathrm{F}; \\ \begin{bmatrix} a_{3}a_{4} \end{bmatrix}: \quad \operatorname{Im}(w_{2} + fw_{1}) = 0, \quad \operatorname{Im}(iw_{1} + \beta w_{2}) = \mathrm{F}. \end{aligned}$$
(7)

Here for brevity sake we'll define by means of $F = \frac{GN - MH}{CN - DM} F'_{x} z'_{z_1}$.

Likewise we'll define through $S(z) = \frac{w_2(z)}{w_1(z)}$. In this case the boundary conditions (7) will be as follows:

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$$\begin{bmatrix} a_1 a_5 a_4 \end{bmatrix}: \qquad \operatorname{Im} S(z) = 0;$$

$$\begin{bmatrix} a_1 a_2 \end{bmatrix}: \qquad \operatorname{Im} \left(\frac{i}{i + \beta f} + \frac{\beta}{i + \beta f} S(z) \right) = 0;$$

$$\begin{bmatrix} a_2 a_3 \end{bmatrix}: \qquad \operatorname{Im} \left(\frac{i + \beta S(z)}{i S(z) - \beta'} \right) = 0;$$

$$\begin{bmatrix} a_3 a_4 \end{bmatrix}: \qquad \operatorname{Im} \left(\frac{i}{i - \beta f} + \frac{\beta}{i - \beta f} S(z) \right) = 0.$$
(8)

Let the function S(z) represent some area S on the lower semi-plane. Then we will have the following condition for each line being a part of the area S contour:

$$\operatorname{Im}\left(\frac{a+b\cdot S(z)}{c+d\cdot S(z)}\right) = 0.$$

It means that each of such lines is being transformed into a real axis, i.e. a straight line by means of a fractional linear transformation. Therefore such lines will be segments of a circle. For this reason the conditions at the straight line $B_1B_5B_4$ (Fig. 4) comply with the conditions at the segment $\begin{bmatrix} a_1a_5a_4 \end{bmatrix}$ (8), and the conditions at the straight lines B_1B_2 and B_3B_4 comply with the conditions at the segments $\begin{bmatrix} a_1a_5a_4 \end{bmatrix}$ (8), and the $\begin{bmatrix} a_3a_4 \end{bmatrix}$ correspondingly. At the segment $\begin{bmatrix} a_2a_3 \end{bmatrix}$ the conditions correspond to the ones at the circle *F*. Let's define the correspondence between the crosshatched area *S* and the lower semi-plane.



Fig. 4. View of the area S

In this case the point B_1 which corresponds to the point a_1 belonging to the boundary of the lower semiplane (Fig. 3) is located at the straight line $B'_2B_5B'_3$, since $S(z)\Big|_{z\to\infty} = \frac{T}{P}$ according to determination of functions $w_1(z)$ and $w_2(z)$, therefore $B_5 \to S(\infty) = \frac{T}{P}$, i.e. the shearing force T can not exceed fP.

The point B_2 aligns with the point of intersection of the straight lines B_1B_2 and $B_2'B_3'$ since $w_1(z)$ and $w_2(z)$ are Cauchy-like integrals values of density of which at $[a_1a_2]$ and $[a_3a_4]$ are proportional due to the conditions t(x) - fP(x) = 0 and t(x) + fP(x) = 0 (Fig. 3). That's why they will have properties of the same magnitude $D_1(z-a_1)^{-\Theta}$ and $D_2(z-a_4)^{-\Theta}$ in the neighborhood of the points a_1 and a_4 (Fig. 3). It means that S(z) in the neighborhood of the points a_1 and a_4 is limited.

Now let's make use of one of the conditions (7) common for the segments $[a_1a_2]$, $[a_2a_3]$, $[a_3a_4]$

$$u_1 + \beta v_2 = \operatorname{Im}(iw_1 + \beta w_2) = 0,$$

but $S(x) = \frac{\omega_2(x)}{\omega_1(x)}$, that's why we have at the segment $[a_1a_4]$;
 $u_1 + \beta v_2 = \operatorname{Im}[(i + \beta \cdot S(x))w_1] = F,$
Im $w_1 = 0$ at $[a_1a_5a_4]$.

In this case for determining of $W_1(x)$ we have a special case of the Riemann-Hilbert problem, thus a uniform problem solution will be formulated as follows:

$$w_1(z) = \exp\left\{-\frac{1}{2\pi} \int_{-e}^{e} \lg\left[\frac{i+\beta \cdot S(\xi)}{-i+\beta \cdot \overline{S}(\overline{\xi})}\right] \frac{d\xi}{\xi-z}\right\} \frac{P(z)}{(z-e)(z+e)}, \quad (9)$$

and a general solution will be as follows:

$$w_2(z) = S(z) \cdot \exp\left\{-\frac{1}{2\pi} \int_{-e}^{e} \lg\left[\frac{i+\beta \cdot S(\xi)}{-i+\beta \cdot \overline{S}(\overline{\xi})}\right] \frac{d\xi}{\xi-z} \right\} \frac{P(z)}{(z-e)(z+e)}, \quad (10)$$

where e - a half of the length of the segment $\lfloor a_1 a_4 \rfloor$.

The functions $W_1(z)$ and $W_2(z)$ may be used for determination of the state of strain below the diamond sphere (tool) only after determining the function S(z).

Let's apply and approximate conformal mapping of the area $\,S$. For this it is necessary to plot the circle $\,F_1$ and F_2 through the points $B_2^\prime B_1$ and $B_3^\prime B_4$ (Fig. 5).

Let the crosshatched area S' approximate the area S. In this case the area S' may be represented as a polygon limited by two coaxial circles and two radii which is then represented as a rectangular which in its turn is mapped at the lower semi-plane [6-7, 9]. In this manner an analytical function transforming the area S' for mapping at the semi-plane is being determined for the definite conditions of penetration of the traveling diamond sphere into the elastic semi-plane. After getting a certain form of the function $S^{\prime}(z)$ it is necessary

to determine the functions $W_1(z)$ and $W_2(z)$ (9-10), which allow defining plastic deformations and strains below the traveling diamond sphere (tool).



Fig. 5. Approximation of the area S boundaries by the circular segments

In order to evaluate the plastic deformations penetration height it is necessary to calculate σ_y - strains along the axis y. Strains of the elastic half-plane are determined as follows:

$$\sigma_{x} = \lambda \theta + 2\mu \varepsilon_{x};$$

$$\sigma_{y} = \lambda \theta + 2\mu \varepsilon_{y};$$

$$\tau_{xy} = 2\mu \gamma_{xy}.$$

where $\theta = \mathcal{E}_x + \mathcal{E}_y$ - volume dilatation, λ and μ - Lame coefficients [6].

The height of the plastically deformed layer was determined based on the following condition

$$\sigma_{y} = \sigma_{0,2} \tag{11}$$

where $\sigma_{0,2}$ - the material conventional yield point.

4. The results of determining the plastically deformed layer height

In the course of calculations of the height of the plastically deformed surface layer for steel of 12X18H9T make there was used an industrial (conventional) compression yield point which makes $\sigma_{0,2} = 196$ MPa. Steel density makes $\rho = 7.63 \cdot 10^3$ kg/m³, the Poisson ration V = 0.3, elastic modulus $E = 2.2 \cdot 10^5$ MPa. In accordance with the works [1-2] the friction coefficient at time of the diamond sphere sliding over steel makes 0.05.

It's apparent that after distinguishing the area with the condition (11) we'll get the boundary of the plastic deformation area. Its maximum penetration into the elastic deformation area was regarded as the plastic deformation (cold work hardening) height.

It's worth noting that the mechanical hardening height in this case was determined with allowance for the tool (diamond sphere) travel speed and elastoplastic characteristics of the semi-plane being processed.

For experimental determination of the plastically deformed layer height there were used the samples surface of which was burnished by the diamond sphere in the following regimens: the diamond sphere radius R_{sph} [R_{sphere}] = 1.5 mm, pressing force of the diamond sphere against the surface being hardened $P_y \subset [50; 100; 150; 180; 200; 250]$ N, the sample rotary speed n = 300 rpm [rounds per minute], the diamond sphere travel encoded along the surface being speed n = 300 rpm [rounds per minute], the

diamond sphere travel speed along the sample rotation axis (feeding speed) $S_o = 0.08$ mm/round. A natural diamond was used as a material for the diamond sphere. Linear travel speed of the diamond sphere was equal to $\mathcal{U} = 1.6$ m/sec.

The solution of the problem under concern was implemented with use of the SciLab software environment. Fig. 6 shows the measuring results.

On Fig. 6 you can see a bar diagram of the burnishing effort effect on the height of the plastically deformed surface layer after finishing the diamond burnishing process.



Fig. 6. The height of the plastically deformed surface layer after diamond burnishing determined by experiment and calculation

As it may be seem from Fig. 6, the height of the plastically deformed layer determined by calculation and by experiment varies from 6 to 20% depending on the burnishing force value. The maximum offset of the hardened layer height is achieved at the maximum load of the diamond sphere (250 N). The error in the received solution could occur due to the plain task. It means that an infinite cylinder having the value of radius equivalent to the diamond sphere radius instead of a sphere is traveling across the elastic semi-plane. For this reason in the course of evaluation of the plastically deformed layer height a bulge of the plastically deformed semi-plane occurring around the traveling sphere along its trajectory is not taken into account (Fig. 7).

In the course of finish machining of the parts under complex load a processing background of the previously applied working methods is of great importance for the mentioned parts performance properties. Quality of the surface layer of the parts subjected to plastic deformation is characterized by a lot of parameters regulated by the national standard (GOST 18296-72. Processing by superficial plastic deformation. Terms and definitions) among which there is the mechanical hardening height (h_H). Due to this the method of determining h_H with the error of $\leq 20\%$ which is based on use of the above dependencies may be applied as a method of initial control of the hight of a layer being deformed by a diamond sphere.



Fig. 7. Diagram of formation of a deformed material bulge in front of the traveling sphere

5. Conclusions

An analytical dependency for determining the hight of a plastically deformed layer after the diamond burnishing process allowing calculating the mentioned parameter with the error not exceeding 20% as compared to the results of experimental investigations was obtained.

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