Two-media boundary layer on a flat plate

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Abstract: The present paper provides a solution to the problem of a flow over a flat semi-infinite plate set at an angle to the horizon, and having a thin liquid film on its surface by external airflow. The film is formed by extrusion of liquid from the porous wall. The paper proposes a mathematical model of a two-media boundary layer flow. The main characteristics of the flow to a zero and a first approximation are determined. A drop of frictional stress is obtained.

Keywords: plate, boundary layer, film, friction

1. INTRODUCTION

Phenomena accompanied by a flow of fluid films are broad and varied. The main experimental and theoretical studies of such films were carried out in the XX century. Among the publications regarding films condensation (Koh et al., 1961), (Tetsu Fujii and Haruo Uehara, 1972), (Mayhew and Aggarwal, 1973), (Stephan, 2006) should be noted. Questions concerning evaporation and boiling are quite well summarized in recent books of (Baehr and Stephan, 2011) and (Ahsan, 2011). Resistance reduction using long-molecular polymers or suspensions is also a well characterized phenomenon. The book (Hugh, 1984) deals with various aspects of resistance reduction. Resistance reduction by addition of polymers to a flow is among the phenomena occurring in the boundary region. Concentrated solutions of additives are continuously introduced into the boundary area where a thin layer of a non-Newtonian liquid is formed in order to increase the effectiveness of the polymers addition.

In this paper we consider a steady-state flow of a non-Newtonian liquid film with a variable thickness on a flat plate under the influence of an incoming air flow, velocity vector of which coincides with the plate plane (Figure 1). Let's suppose that the plate is at [alpha] angle to the horizon. Then a liquid flows due to gravity and friction on the outer surface of the film.

Let's define the influence of the film on the friction value in the boundary layer. In general the problem is conjugate, including a problem of a film flow (internal problem) and a problem of the boundary layer of incoming air (external problem). To solve the dual problem a method of successive approximations is used. The method consists in the fact that the external and internal problems can be solved separately, but logically or iteratively.

At each new approximation the inner problem is solved with regard to the friction resulting from the external problem, the solution of which, in turn, takes into account speed at boundary surface, obtained in the previous approximation from the interior problem. Thus, the iterative process continues until the speed and, consequently, the friction at the phase interface change little from iteration to iteration.

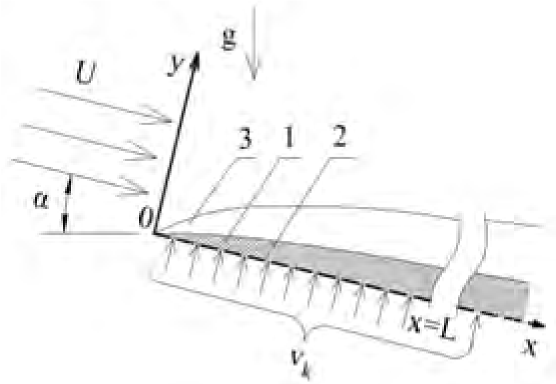


Fig. 1. Flow diagram. 1 - film, 2 - plate, 3 - boundary layer, U – incident flow velocity, g – free fall acceleration, [alpha] - angle of inclination of the plate to the horizontal, x and y - Cartesian axis, L - length of the plate, vk – mass addition speed

Let the film thickness [delta] ≈ 10-4 m, and imposition of the film changes geometry of the plate slightly. Therefore, to solve the external problem film surface curvature can be neglected.

2. METHOD

The task of the boundary layer. A mathematical formulation of the exterior problem includes motion and continuity equations in the approximation of the boundary layer (index "3" corresponds to the boundary layer, index "1" - to the film)

$$u_3 \frac{\partial u_3}{\partial x} + v_3 \frac{\partial u_3}{\partial y} = \nu_3 \frac{\partial^2 u_3}{\partial y^2} \tag{1}$$

$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} = 0 \tag{2}$$

at boundary conditions:

$$y = 0,\ u_3 = u_1(x),\ v_3 = v_1(x);\ y = \delta_3,\ u = U. \tag{3}$$

Boundary layer equations are not applicable in the intermediate vicinity of the plate edge. Therefore, in the vicinity of a point x=0 initial (marked by index «s») velocity profiles and boundary layer thickness are defined.

$$x = 0,\ u_3 = u_s,\ v_3 = v_s,\ \delta_3 = \delta_s \tag{4}$$

Let's introduce non-dimentional variables and save their former notations

$$x = \frac{x}{L},\ y = \frac{y}{\delta_3},\ \delta_3 = \frac{\delta_3}{L},\ u_3 = \frac{u_3}{U},\ v_3 = \frac{v_3}{U} \tag{5}$$

Substituting (1.5) into the equation of motion (1.1), we obtain

$$u_3 \left[\frac{\partial u_3}{\partial x} - \frac{y\delta_3'}{\delta_3} \frac{\partial u_3}{\partial y} \right] + \frac{v_3}{\delta_3} \frac{\partial u_3}{\partial y} = \frac{1}{\mathrm{Re}_3\, \delta_3^2} \frac{\partial^2 u_3}{\partial y^2}, \tag{6}$$

where $\mathrm{Re}_3 = \frac{U\,L}{\nu_3}$ - Raynolds number, u, v - velocity projections on x and y axes correspondingly, δ - thickness, ν - kinematic viscosity, a stroke means derivative with respect to x.

Continuity equation (2) will be as follows

$$\frac{\partial u_3}{\partial x} - \frac{y\delta_3'}{\delta_3} \frac{\partial u_3}{\partial y} + \frac{1}{\delta_3} \frac{\partial v_3}{\partial y} = 0, \tag{7}$$

boundary conditions (3) and (4) will be written over as follows

$$y = 0,\ u_3 = u_1 / U,\ v_3 = v_1 / U;\ y = 1,\ u_3 = 1, \tag{8}$$

$$x = 0,\ u_3 = u_s / U,\ v_3 = v_s / U,\ \delta_3 = \delta_s / L \tag{9}$$

The equation of motion (1) and continuity (2) correspond to the classical formulation (Blasius problem), and the boundary condition at infinity is replaced by an approximate one, namely, instead of $y \to \infty,\ u_3 \to U$ (3) is used. Equations (6) and (7) contain three unknown functions u_3, v_3, δ_3. To close the system one more equation is added, and for this purpose an equation of motion over the boundary layer thickness is integrated.

$$\begin{aligned} &\int_0^1 u_3 \frac{\partial u_3}{\partial x} dy - \frac{\delta_3'}{2\,\delta_3}\left[1 - \int_0^1 u_3^2 dy\right] + \\ &+ \frac{v_3}{\delta_3}\int_0^1 \frac{\partial u_3}{\partial y} dy + \frac{1}{\mathrm{Re}_3\, \delta_3^2} \left.\frac{\partial u_3}{\partial y}\right|_{y=0} = 0. \end{aligned} \tag{10}$$

During the integration some members are dropped due to the use of the condition of smooth closing of the longitudinal velocity with the incident flow velocity $\partial u_3 / \partial y = 0$ at $y = 1$, which is not contained explicitly in the classical formulation, but was introduced in addition to the boundary condition (3). Thus, the third linearly independent equation is obtained (10).

The problem for a flowing liquid film. In the approximation of the boundary layer theory let's write the equation of motion and continuity for incompressible fluid

$$\mathrm{u}_1 \frac{\partial \mathrm{u}_1}{\partial \mathrm{x}} + \mathrm{v}_1 \frac{\partial \mathrm{u}_1}{\partial \mathrm{y}} = g \sin\alpha + \nu_1 \frac{\partial^2 \mathrm{u}_1}{\partial \mathrm{y}^2}, \tag{11}$$

$$\frac{\partial \mathrm{u}_1}{\partial \mathrm{x}} + \frac{\partial \mathrm{v}_1}{\partial \mathrm{y}} = 0. \tag{12}$$

Let's establish a condition of adhesion and mass addition at the plate $\mathrm{y} = 0,\ \mathrm{u}_1 = 0,\ \mathrm{v}_1 = v_k$, on the film surface at $\mathrm{y} = \delta_1$ - friction obtained by solving the boundary layer problem $\tau_1 = \mu_1 \frac{\partial \mathrm{u}_1}{\partial \mathrm{y}}$. Let's supplement the equation of motion with a kinematic condition at the interface boundary

$$\mathrm{v}_1 = \mathrm{u}_1 \delta_1' \tag{13}$$

Let's introduce non-dimensional variables for the film (their notation are saved)

$$\begin{gathered} x = \frac{\mathrm{x}}{L},\ y = \frac{\mathrm{y}}{\delta^*},\ \delta_1 = \frac{\delta_1}{\delta^*},\ u_1 = \frac{\mathrm{u}_1}{u^*}, \\ v_1 = \frac{\mathrm{v}_1}{v_k},\ \tau_1 = \frac{\delta^*}{\mu_1\ u^*}\tau_1. \end{gathered} \tag{14}$$

Taking into account (14) the motion equation (11) will be as follows

$$\varepsilon \left(u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} \right) = \frac{\nu_1}{\delta^* u^*} \frac{\partial^2 u_1}{\partial y^2} + \frac{\delta^* g \sin\alpha}{u^{*2}}$$

where δ^* and u^* - thickness and longitudinal velocity scales correspondingly, $\varepsilon = \delta^* / L$ - small parameter. To determine the scale of thickness of δ^* and the longitudinal velocity u^* of the film let's use the balance of gravity forces and viscous stress for a free draining film on a vertical wall (Baehr and Stephan, 2011). Then from the equation (11), scales connection has the following correlation

$$\frac{\nu_1}{\delta^* u^*} = \frac{\delta^* g}{u^{*2}} \tag{15}$$

From the continuity equation (12) we obtain the scales connection $u^* = v_k L / \delta^*$, and considering (15) we will have $\delta^* = \sqrt[3]{\frac{v_k\ L\ \nu_1}{g}}$ $, u^* = \sqrt[3]{\frac{v_k{}^2\ L^2\ g}{\nu_1}}$.

Zero-order approximation. Let's represent the unknown functions in expanded forms in powers of a small parameter, neglecting items of second and higher orders of smallness (hereinafter in the section subscripts refer to the order of approximation and variables without a subscript correspond to the film) $u = u_0 + \varepsilon u_1$, $v = v_0 + \varepsilon v_1$, $\delta = \delta_0 + \varepsilon \delta_1$, then in a zero approximation the boundary value problem for the longitudinal velocity has the form

$$\frac{\partial^2 u_0}{\partial y^2} = -\sin\alpha, u_0 \Big|_{y=0} = 0, \frac{\partial u_0}{\partial y}\Big|_{y=\delta} = \tau \tag{16}$$

Let's write the integral (16)

$$u_0 = \delta_0{}^2 \left(\frac{y}{\delta_0} - \frac{1}{2}\left(\frac{y}{\delta_0}\right)^2 \right) \sin\alpha + \tau\, y \tag{17}$$

Let's define the transverse velocity profile, integrating the continuity equation, taking into account (2.7) and the boundary condition $y = 0, v_0 = 1$

$$v_0 = 1 - \left(\tau + \delta_0 \sin\alpha\right)' \frac{y^2}{2} \tag{18}$$

Substituting the non-dimensional variables (14) and then the expression (17) and (18) into (13), after simple transformations, we obtain the equation for δ_0

$$2\,\delta_0^{\,3} \sin\alpha + 3\,\delta_0^{\,2}\,\tau - 3\,x = 0, \delta_0\left(0\right) = 0\,.$$

First approximation. Let's write the equations of motion and continuity

$$\frac{\partial^2 u_1}{\partial y^2} = \mathrm{Re}_1 \left(u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} \right), \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0,$$

which are supplemented with boundary conditions at the plate $y = 0, u_1 = 0, v_1 = 0,$ and on the film surface

$$y = \delta_0 + \varepsilon\delta_1, \frac{\partial u_1}{\partial y} = 0 \tag{19}$$

where $\mathrm{Re}_1 = \delta^* u^* / \nu_1$ - Raynolds number.

Let's expand the function in a series in the neighborhood of [delta0] with the first order of accuracy

$$\begin{aligned} u\left(\delta_0 + \varepsilon\delta_1\right) &= u\left(\delta_0\right) + \varepsilon\delta_1 \frac{\partial u}{\partial y}\left(\delta_0\right) = \\ &= u\left(\delta_0\right) + \varepsilon u_1\left(\delta_0\right) + \varepsilon\delta_1 \left[\frac{\partial u_0}{\partial y}\left(\delta_0\right) + \varepsilon \frac{\partial u_1}{\partial y}\left(\delta_0\right) \right] \end{aligned} \tag{20}$$

Let's rewrite (19) substituting the expansion (20) and making the coefficients equal at [epsilon]

$$y = \delta_0, \frac{\partial u_1}{\partial y} = \delta_1 \sin\alpha\,.$$

Let's substitute the unknown functions expanded in a series with the first order of accuracy in the kinematic condition)

$$\begin{aligned} y = \delta_0, \left(v_0 + \varepsilon v_1\right) + \varepsilon\delta_1 \left(\frac{\partial v_0}{\partial y} + \varepsilon \frac{\partial v_1}{\partial y} \right) &= \\ = \left[\left(u_0 + \varepsilon u_1\right) + \varepsilon\delta_1 \left(\frac{\partial u_0}{\partial y} + \varepsilon \frac{\partial u_1}{\partial y} \right) \right] \left(\delta_0 + \varepsilon\delta_1\right) \end{aligned} \tag{21}$$

Equating the coefficients at [epsilon] in (21), and making some simple transformations, we obtain the final form of the kinematic condition

$$y = \delta_0, v_1 - u_1\delta_0' = \left[\left(\delta_0\tau + \frac{\delta_0^{\,2}}{2} \right) \delta_1 \right]'$$

Omitting the procedure of unknown functions finding in the first approximation, which coincides with the zero-order approximation, let's write the solution ,

$$\begin{aligned} u_1 = \mathrm{Re}_1 \Bigg(&\left(p\frac{y^2}{2} - \frac{y^3}{6} \right) + \left(p^2\right)' \frac{y^4}{48} + \\ &+ \left[\frac{\delta_1 \sin\alpha}{\mathrm{Re}_1} - \left(p\delta_0 - \frac{\delta_0^{\,2}}{2} \right) - \frac{\delta_0^{\,3}}{12}\left(p^2\right)' \right] y \Bigg) \end{aligned}$$

$$v_1 = -\mathrm{Re}_1 \left(\frac{p' y^3}{6} + \frac{\left(p^2\right)'' y^5}{240} + \right.$$

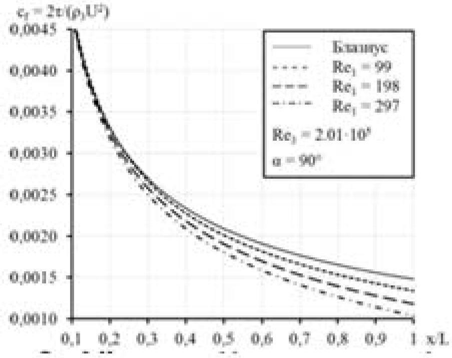
$$\left. + \left[\frac{\delta_1' \sin\alpha}{\mathrm{Re}_1} - \left(p\delta_0 - \frac{\delta_0^2}{2} \right)' - \frac{1}{12}\left(\delta_0^3 \left(p^2\right)' \right)' \right] \frac{y^2}{2} \right)$$

$$\delta_1 = \frac{\mathrm{Re}_1}{q} \left(\frac{3}{40}\left(p^2\right)' \delta_0{}^4 - \frac{5\,\delta_0^3}{12} + \frac{2}{3} p\delta_0^2 \right),$$

Where the notation $p = \tau + \delta_0 \sin\alpha, q = p + \delta_0 + \tau$ is introduced.

3. RESULTS

Fig. 2 and 3 show graphs of variation of the friction coefficient $c_f = 2\tau / \left(\rho_3 U^2\right)$ and the longitudinal velocity u/U at the interface along the plate length at different Re_1 and constant [alpha] and Re_3. Figure 2 also shows the friction distribution along the plate length calculated by the Blasius formula (Schlichting, 1969). Fig. 4 and 5 show the variation of the interfacial friction coefficient and longitudinal velocity as a function of [alpha]. As expected, with increase of Re_1 and inclination of the plate, the velocity at the interface increases, and the friction decreases. In fig. 2 - 5 all values are provided taking into account the first approximation. As for unpublished results we should note the influence of Re_3 on the process characteristics: with an increase of this parameter from its minimum values to the critical ones ($\mathrm{Re}_3 \geq 3 \cdot 10^5$) the impact of the film on the frictional resistance decreases. Thus, when the frictional resistance $\mathrm{Re}_3 = 6.71 \cdot 10^4$ in the presence of the film frictional resistance may many times vary depending on the angle of inclination and Re_1 , whereas at $\mathrm{Re}_3 = 4.7 \cdot 10^5$ the maximum difference will be no greater than 15%.



/Блазиус – Blasius/

Fig. 2 Variation of friction coefficient at the phase contact area along the plate length at $\mathrm{Re}_3 = 2.01 \cdot 10^5$, $\alpha = 90°$ and different Re_1.

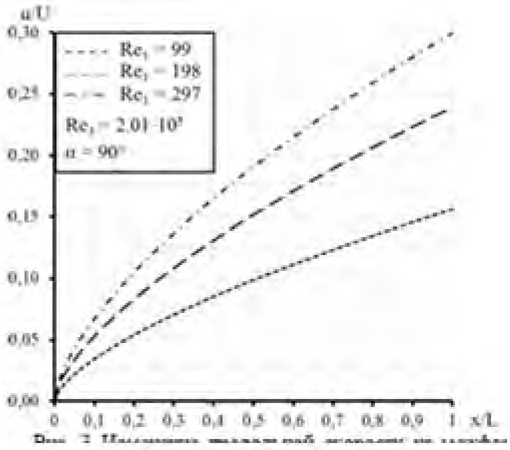


Fig. 3 Variation of longitudinal velocity at the phase contact area along the plate length at $\mathrm{Re}_3 = 2.01 \cdot 10^5$, $\alpha = 90°$ and different Re_1 .

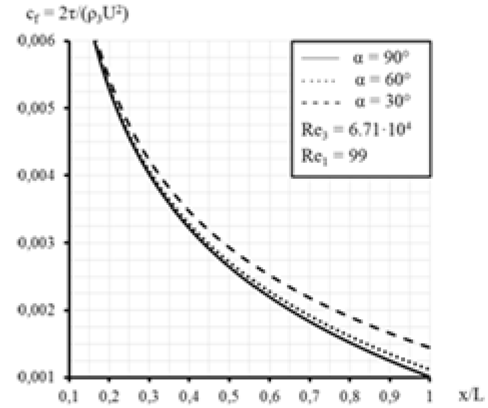


Fig. 4 Variation of frictional coefficient at the phase contact area along the plate length at $\mathrm{Re}_3 = 6.71 \cdot 10^4$, $\mathrm{Re}_1 = 99$ and different α .

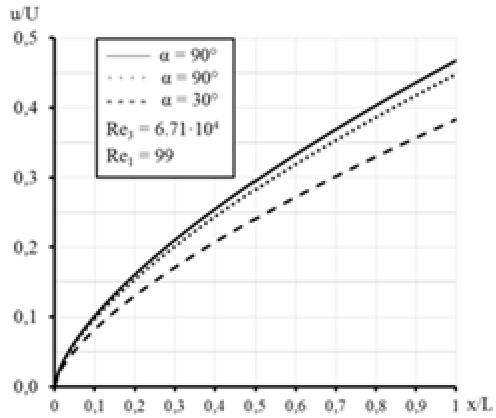


Fig. 5 Variation of longitudinal velocity at the phase contact area along the plate length at $\mathrm{Re}_3 = 6.71 \cdot 10^4$, $\mathrm{Re}_1 = 99$ and different α .

4. DISCUSSION

The system of integro-differential equations (6), (7) an (10) of the boundary layer is solved using a numerically implicit scheme with the help of a method of delayed coefficients. In order to make sure that the selected method is correct the said system with boundary conditions of adhesion and impermeability was tested using analytical solution of Blasius (Schlichting, 1969). The maximum difference between the estimated value of frictional resistance and the exact solution is not more than 5% in the immediate vicinity of the plate beginning and monotonically decreases to a value less than 0.5% at a length L. The difference between the velocity profiles does not exceed 1% over the entire length of the plate. Thus, the system of equations (6), (7) and (10) can be used to calculate the characteristics of the boundary layer. In solving the dual problem analytical profiles of Blasius can be used as the initial approximations u_s, v_s , however linear profiles deliver good final results as well.

It is known that a liquid film on the surface of a body blown over by an airflow changes characteristics of the boundary layer. The works of K.K. Fedyaevskiy (Fedyaevskiy, 1940), (Fedyaevskiy, 1943) and L.G. Loitsianskiy (Loitsyanskiy, 1942) present two-media boundary layers, when a fluid of one type is present or added in a wall-adjacent region and the whole system is placed in an incoming flow of a liquid of another type. In the work of (Loitsyanskiy, 1942) a question regarding assessment of a possible gain in resistance due to introduction of a liquid to the boundary layer (mixing with the surrounding liquid due to diffusion and convection), and counting of the required consumption of the liquid is considered.

In the papers of (Shakhov et al., 2012) and (Shakhov et al., 2013) two-media boundary layer when the media do not mix are considered. It is assumed that applied through a streamlined surface medium is enclosed in a thin region where of the longitudinal velocity distribution can be considered linear. In (Shakhov et al., 2012) a method of integral relations for the case of flow over a flat plate is used, whereas in (Shakhov et al., 2013) a method of finite differences for an arbitrary body is developed.

5. CONCLUSION

In general, the solution to the dual problem depends on three parameters - $\mathrm{Re}_1, \mathrm{Re}_3$ and angle α . The solution to the problem of a laminar boundary layer on a flat plate is limited by $\mathrm{Re}_3 < 3\cdot 10^5 \div 3\cdot 10^6$ and laminar flow of a liquid film does not exceed $\mathrm{Re} = \dfrac{\delta_1(L) < \mathrm{u} >_L}{\nu_1} < 400,$

where $< \mathrm{u} >_L = \dfrac{1}{\delta_1(L)} \int\limits_0^{\delta_1(L)} \mathrm{u}_1(L, y)dy$ - - average longitudinal velocity of the liquid film on the length L (Kutateladze and Styrikovich, 1976) (Schlichting, 1969). Therefore, the proposed mathematical model can be used with the above restrictions. In fig. 1 – 5 $\mathrm{Re} < 400$.

The analysis of the results allows to conclude that the liquid film on the surface of a body reduces the frictional resistance. For bluff bodies friction contribution to the overall resistance of the body is small. On the contrary, for streamlined bodies, friction plays a defining role. Consequently, when calculating the aerodynamic characteristics of a streamlined body the presence of a film on the body surface should be taken into account.

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