Hankel Tranform Method for Solving Axisymmetric Elasticity Problems of Circular Foundation on Semi-infinite Soils

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Abstract – **The Hankel transform method was implemented in this paper to obtain general solutions for stress and displacement fields in homogeneous, isotropic linear elastic, semi-infinite soil subject to uniformly distributed axisymmetric load distribution over a circular area on the surface. The work adopted a stress based formulation. Hankel transformation of the biharmonic stress compatibility equation was done to obtain bounded stress functions for the elastic half space problem. Hankel transformations were also applied to the Love expression to obtain the stresses and displacements in Hankel transform space. Enforcement of boundary conditions yielded the unknown constants of the stress functions for the particular problem considered. Inversion of the Hankel transform expressions for the stresses and displacements yielded the general expressions for stresses and displacements for the semi-infinite linear elastic soil under uniformly distributed load over a circular area on the surface. The solutions obtained were identical with solutions in literature obtained using Boussinesq stress functions.**

Keywords: *Hankel transform method, elastic half space problem, axisymmetric elasticity problem, displacement fields, stress compatibility equation.*

I. INTRODUCTION

The problem of the determination of stress and displacement fields due to uniformly distributed load applied to the surface of a linear elastic soil mass assumed isotropic, homogeneous and of semi-infinite extent belongs to the classical mathematical theory of elasticity [1]. Such problems have extensive applications in the analysis and design of structural footings or foundation structures [2, 3]. For uniformly distributed load over a circular loaded area on the elastic half space, the problem is axisymmetrical where the center of the loaded area is the vertical axis of symmetry. The problem can be formulated and solved by simultaneous consideration of the three differential equations of equilibrium, the six material constitutive laws, the six kinematic relations, the compatibility equations, and the traction and deformation boundary conditions. The number and complexity of the governing (field) equations of elasticity theory for a rigorous mathematical formulation and solution are quite unwieldly and involve extensive and intensive mathematically rigorous techniques for analysis and solution [4].

Two basic methods commonly applied in the formulation of elasticity problems are the displacement based or displacement method and stress based or stress method [5, 6, 7]. A third method which is not commonly used is the mixed method.

The displacement based methods involve a reformulation of the three fundamental requirements of the differential equations of equilibrium, the generalised stress-strain laws and the strain-displacement relations such that the Cartesian components of displacement become the only unknown primary variables. This has the advantage of achieving a reduction in the number of governing equations from a set of fifteen equations in a three dimensional formulation to a set of three coupled equations. The displacement formulation was presented by Navier and Lamé.

In stress based formulations, the system of fifteen differential equations involving equilibrium, kinematics and material constitutive laws are simultaneously reformulated such that the strains and displacements are eliminated and the Cauchy stresses become the unknown primary variables [5, 6, 7]. The advantage is that the governing equations are reduced for three dimensional problems in Cartesian coordinates from fifteen to six equations in terms of the three components of normal stresses and the three components of shear stresses. Stress based formulations of elasticity problems were presented by Michell and Beltrami. In a mixed formulation, the governing equations or field equations are re-formulated such that some components of stress and some displacement components become the unknown primary variables.

This study uses the stress formulation method. The analytical simplifications provided by the reformulation of the general three dimensional elasticity problem have led to the derivations of stress and displacement functions that are apriori solutions of the governing equations in a stress based and displacement based formulation [6, 7, 8].

Such stress and displacement functions further simplify the solutions of elasticity problems to finding appropriate stress and displacement functions that satisfy the boundary conditions of the given problem. This reduces the dimensionality of the general problem of the mathematical theory of elasticity. Stress functions derived such that they are functions of the space coordinate variables that automatically satisfy all the differential equations of equilibrium and the strain compatibility equations include Airy's stress function, Morera stress functions, Maxwell stress functions, Michell stress functions and Love [8] stress functions.

Elasticity problems of semi-infinite soil media under boundary loads have been studied by Onah et al [9]; Ike et al [10]; Nwoji et al [11]; Ike et al [12]; Onah et al [13]; Nwoji et al [14] and Onah et al [15].

II. RESEARCH AIM AND OBJECTIVES

The research aim is to implement the Hankel transform method for solving the axisymmetric elasticity problem of circular foundation on semi-infinite soil. The objectives are:

(i) to obtain bounded stress functions that are solutions to the stress compatibility equation for elastic half space by application of the Hankel transformation.

(ii) to apply the Hankel transformation to the Love stress functions in order to obtain the stresses $\sigma_{zz}, \tau_{rz}, \sigma_{rr}, \sigma_{\theta\theta}$ in the Hankel transform space corresponding to the bounded stress functions obtained.

(iii) to obtain displacement field components corresponding to the bounded stress functions in the Hankel transform space.

(iv) to apply the boundary conditions and obtain the unknown constants of the stress function.

(v) to apply the inverse Hankel transformation to the displacements and stresses in the Hankel transform space in order to obtain the displacements and stresses in the physical domain space variables.

III. GOVERNING EQUATIONS

The governing equations of axisymmetric elasticity problems of circular foundations of radius $r = R$ subject to a uniformly distributed load of intensity p over the foundation area were formulated in stress terms as [16]:

$$\nabla^4\phi(r,z) = \nabla^2\nabla^2\phi(r,z) = 0 \tag{1}$$

$$0 \le r \le \infty, \quad 0 \le z \le \infty$$

subject to the boundary conditions:

$$\sigma_{zz}(r, z=0) = -p \qquad 0 \le r \le R \tag{2}$$

$$\sigma_{zz}(r, z=0) = 0 \qquad r > R \tag{3}$$

$$\tau_{rz}(r, z=0) = 0 \qquad 0 \le r \le \infty \tag{4}$$

where r is the radial coordinate, z is the depth (vertical) coordinate, $\sigma_{zz}(r,z)$ is the vertical stress distribution at the point (r, z) in the semi-infinite elastic soil, τ_{rz} is the shear stress distribution at the point (r, z) in the semi-infinite elastic soil, ∇^2 is the Laplacian given by the scalar differential operator:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{5}$$

and ∇^4 is the biharmonic operator defined as:

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) \tag{6}$$

$\phi(r,z)$ is the stress function.

Love defined the stress function $\phi(r,z)$ for axisymmetric elasticity problems as follows:

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{7}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{8}$$

$$\sigma_{rr} = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \quad (9)$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right] \quad (10)$$

where μ is the Poisson's ratio of the soil, σ_{rr} is the radial stress distribution in the semi-infinite elastic soil, $\sigma_{\theta\theta}$ is the circumferential (hoop) stress in the semi-infinite soil.

The displacement field components are given from the stress functions $\phi(r,z)$ as follows:

$$u = -\frac{1}{2G}\frac{\partial^2\phi}{\partial r\partial z} = -\left(\frac{1+\mu}{E}\right)\frac{\partial^2\phi}{\partial r\partial z} \quad (11)$$

$$w = \frac{1}{2G}\left(2(1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right) \quad (12)$$

$$w = \left(\frac{1+\mu}{E}\right)\left(2(1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right) \quad (13)$$

where G is the shear modulus of the soil, E is the Young's modulus of elasticity, G is related to E by

$$G = \frac{E}{2(1+\mu)} \quad (14)$$

IV. METHODOLOGY

Application of the Hankel transforms to the governing equations

The Hankel transform of a function $f(r)$ is denoted by $F_n(k)$, and defined as the integral transform:

$$F_n(k) = \int_0^\infty r\, f(r) J_n(kr)\,dr \quad (15)$$

where $J_n(kr)$ is the Bessel function of order n, and k is the parameter of the transform.

By inversion,

$$f(r) = \int_0^\infty F_n(k) J_n(kr)\,k\,dk \quad (16)$$

The governing stress compatibility equation is given explicitly by:

$$\nabla^4\phi(r,z) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \quad (17)$$

Application of the Hankel transform to the governing equation yields:

$$\int_0^\infty (\nabla^4\phi(r,z)) J_0(kr) r\,dr = 0 \quad (18)$$

or

$$\int_0^\infty \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}\right) J_0(kr) r\,dr = 0 \quad (19)$$

Let $\beta = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}$ (20)

Then

$$\int_0^\infty \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\beta r\, J_0(kr)\,dr = 0 \quad (21)$$

$$\int_0^\infty \left(\frac{\partial^2\beta}{\partial r^2} + \frac{1}{r}\frac{\partial\beta}{\partial r} + \frac{\partial^2\beta}{\partial z^2}\right) r\, J_0(kr)\,dr = 0 \quad (22)$$

$$\int_0^\infty \left(\frac{\partial^2\beta}{\partial r^2} + \frac{1}{r}\frac{\partial\beta}{\partial r}\right) r\, J_0(kr)\,dr + \int_0^\infty \frac{\partial^2\beta}{\partial z^2} r J_0(kr)\,dr = 0 \quad (23)$$

$$\int_0^\infty \left(\frac{\partial^2 \beta}{\partial r^2} + \frac{1}{r}\frac{\partial \beta}{\partial r} \right) r\, J_0(kr)dr + \frac{\partial^2}{\partial z^2}\int_0^\infty \beta r J_0(kr)dr = 0 \tag{24}$$

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r(J_0(kr))dr + \int_0^\infty \left(r\frac{\partial^2 \beta}{\partial r^2} + \frac{\partial \beta}{\partial r} \right) J_0(kr)dr = 0 \tag{25}$$

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r(J_0(kr))dr + \int_0^\infty \frac{\partial}{\partial r}\left(r\frac{\partial \beta}{\partial r} \right) J_0(kr)dr = 0 \tag{26}$$

Integration by parts of the second integral in Equation (26) yields:

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r J_0(kr)dr + \left[r\frac{\partial \beta}{\partial r} J_0(kr) \right]_0^\infty - \int_0^\infty r\frac{\partial \beta}{\partial r}\left(\frac{\partial}{\partial r} J_0(kr) \right) dr = 0 \tag{27}$$

$\frac{\partial \beta}{\partial r}$ is assumed to be finite for all values of r, hence

$$\left[r\frac{\partial \beta}{\partial r} J_0(kr) \right]_0^\infty = 0 \tag{28}$$

$$\frac{\partial}{\partial r} J_0(r) = -J_1(r) \tag{29}$$

$$\frac{\partial}{\partial r} J_0(kr) = -kJ_1(kr) \tag{30}$$

$$\frac{\partial}{\partial r}(rJ_1(kr)) = rkJ_0(kr) \tag{31}$$

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r J_0(kr)dr - \int_0^\infty r\frac{\partial \beta}{\partial r}(-kJ_1(kr))dr = 0 \tag{32}$$

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r J_0(kr)dr + k\int_0^\infty r\frac{\partial \beta}{\partial r} J_1(kr)dr = 0 \tag{33}$$

Integrating by parts,

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r J_0(kr)dr + k\left\{ \left[\beta\, J_1(kr) \right]_0^\infty - \int_0^\infty \beta \frac{\partial}{\partial r}(rJ_1(kr))dr \right\} = 0 \tag{34}$$

$$\beta\, J_1(kr)\big]_0^\infty = 0 \tag{35}$$

$$\frac{\partial^2}{\partial z^2}\int_0^\infty \beta r\, J_0(kr)dr - k^2\int_0^\infty \beta J_0(kr) r\, dr = 0 \tag{36}$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right)\int_0^\infty \beta J_0(kr) r\, dr = 0 \tag{37}$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right)\int_0^\infty \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right) J_0(kr) r\, dr = 0 \tag{38}$$

By a similar procedure, we obtain

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right)\left(\frac{\partial^2}{\partial z^2} - k^2 \right)\int_0^\infty r\phi(r,z) J_0(kr)\, dr = 0 \tag{39}$$

Thus,

$$\int_0^\infty \nabla^4 \phi(r,z) J_0(kr) r dr = \left(\frac{\partial^2}{\partial z^2} - k^2 \right)^2 \int_0^\infty \phi(r,z) r J_0(kr) dr \tag{40}$$

Let $\bar{\phi}(k,z)=\int_0^{\infty} r\phi(r,z)J_0(kr)dr$ (41)

where $\bar{\phi}(k,z)$ is the Hankel transform of $\phi(r,z)$

Then, we have

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)^2\bar{\phi}(k,z)=0 \quad (42)$$

Solving, by using the trial function method, let

$$\bar{\phi}(k,z)=\exp mz \quad (43)$$

$$\left(\frac{d^2}{dz^2}-k^2\right)^2 e^{mz}=0 \quad (44)$$

$$(m^2-k^2)^2 e^{mz}=0 \quad (45)$$

The characteristic polynomial is

$$(m^2-k^2)^2=0 \quad (46)$$

$m=\pm k$ (twice) (47)

The solution is

$$\bar{\phi}(x,z)=(c_1+c_2z)\exp(-kz)+(c_3+c_4z)\exp kz \quad (48)$$

where c_1, c_2, c_3, c_4 are the integration constants

For bounded solutions of the stress fields $\bar{\phi}(k,z)$ should be finite as $z\to\infty$.

Hence $c_3 = c_4 = 0$ (49)

and

$$\bar{\phi}(k,z)=(c_1+c_2z)\exp(-kz) \quad (50)$$

Applying the Hankel transforms to the Love stress functions for σ_{zz} and τ_{rz}, we obtain:

$$\bar{\sigma}_{zz}=\int_0^{\infty}\frac{\partial}{\partial z}\left((2-\mu)\nabla^2\phi-\frac{\partial^2\phi}{\partial z^2}\right)rJ_0(kr)dr \quad (51)$$

where $\bar{\sigma}_{zz} =$ Hankel transform of σ_{zz}

$$\bar{\sigma}_{zz}=(2-\mu)\int_0^{\infty}\frac{\partial}{\partial z}\nabla^2\phi rJ_0(kr)dr-\int_0^{\infty}\frac{\partial^3\phi}{\partial z^3}rJ_0(kr)dr \quad (52)$$

$$\bar{\sigma}_{zz}=(2-\mu)\int_0^{\infty}\nabla^2\frac{\partial\phi}{\partial z}rJ_0(kr)dr-\int_0^{\infty}\frac{\partial^3\phi}{\partial z^3}rJ_0(kr)dr \quad (53)$$

$$\bar{\sigma}_{zz}=(2-\mu)\cdot -k^2\int_0^{\infty}\frac{\partial\phi}{\partial z}rJ_0(kr)dr-\frac{\partial^3}{\partial z^3}\int_0^{\infty}\phi rJ_0(kr)dr$$

$$+\,(2-\mu)\frac{\partial^2}{\partial z^2}\int_0^{\infty}\frac{\partial\phi}{\partial z}rJ_0(kr)dr \quad (54)$$

$$\bar{\sigma}_{zz}=-(2-\mu)k^2\frac{\partial\bar{\phi}}{\partial z}+(2-\mu)\frac{\partial^3\bar{\phi}}{\partial z^3}-\frac{\partial^3\bar{\phi}}{\partial z^3} \quad (55)$$

$$\bar{\sigma}_{zz}=-(2-\mu)k^2\frac{\partial\bar{\phi}}{\partial z}+(1-\mu)\frac{\partial^3\phi}{\partial z^3} \quad (56)$$

Similarly, for the shear stress,

$$\bar{\tau}_{rz}=\int_0^{\infty}\frac{\partial}{\partial r}\left((1-\mu)\nabla^2\phi-\frac{\partial^2\phi}{\partial z^2}\right)rJ_0(kr)dr \quad (57)$$

$$\bar{\tau}_{rz}=(1-\mu)\int_0^{\infty}\frac{\partial}{\partial r}\nabla^2\phi rJ_0(kr)dr-\int_0^{\infty}\frac{\partial}{\partial r}\frac{\partial^2\phi}{\partial z^2}rJ_0(kr)dr \quad (58)$$

$$\frac{\partial}{\partial r}\left\{\int_0^\infty (1-\mu)\nabla^2\phi r J_0(kr)dr - \int_0^\infty \frac{\partial^2\phi}{\partial z^2} r J_0(kr)dr\right\} = \overline{\tau}_{rz} \quad (59)$$

$$\frac{\partial}{\partial r}\left\{(1-\mu)\int_0^\infty \nabla^2\phi r J_0(kr)dr - \frac{\partial^2}{\partial z^2}\int_0^\infty \phi r J_0(kr)dr\right\} = \overline{\tau}_{rz} \quad (60)$$

$$\frac{\partial}{\partial r}\left\{(1-\mu)\left(\frac{\partial^2}{\partial z^2} - k^2\right)\overline{\phi} - \frac{\partial^2}{\partial z^2}\overline{\phi}\right\} = \overline{\tau}_{rz} \quad (61)$$

$$\frac{\partial}{\partial r}\left\{(1-\mu)\frac{\partial^2\overline{\phi}}{\partial z^2} - \frac{\partial^2\overline{\phi}}{\partial z^2} - (1-\mu)k^2\overline{\phi}\right\} = \overline{\tau}_{rz} \quad (62)$$

$$\frac{\partial}{\partial r}\left\{-\mu\frac{\partial^2\overline{\phi}}{\partial z^2} - (1-\mu)k^2\overline{\phi}\right\} = \overline{\tau}_{rz} \quad (63)$$

$$-\frac{\partial}{\partial r}\left\{(1-\mu)k^2\overline{\phi} + \mu\frac{\partial^2\overline{\phi}}{\partial z^2}\right\} = \overline{\tau}_{rz} \quad (64)$$

$$\overline{\sigma}_{zz} = \int_0^\infty \sigma_{zz} r J_0(kr)dr = \int_0^R \sigma_{zz} r J_0(kr)dr \quad (65)$$

$$\overline{\sigma}_{zz} = \int_0^R p r J_0(kr)dr = p\int_0^R r J_0(kr)dr \quad (66)$$

$$(1-\mu)\frac{\partial^3\overline{\phi}}{\partial z^3} - (2-\mu)k^2\frac{\partial\overline{\phi}}{\partial z} = -p\int_0^R r J_0(kr)dr \quad (67)$$

$$(1-\mu)\frac{\partial^3\overline{\phi}}{\partial z^3} - (2-\mu)k^2\frac{\partial\overline{\phi}}{\partial z} = \frac{-pRJ_1(kR)}{k} \quad (68)$$

Let $g = (1-\mu)\nabla^2\phi - \dfrac{\partial^2\phi}{\partial z^2}$ (69)

Then from the Hankel transform of τ_{rz},

$$\overline{\tau}_{rz} = \int_0^\infty \frac{\partial g}{\partial r} r J_1(kr)dr \quad (70)$$

Integrating by parts,

$$\overline{\tau}_{rz} = \left[grJ_1(kr)\right]_0^\infty - \int_0^\infty g(r,k)(rJ_1(kr))'dr \quad (71)$$

The first term vanishes when $g(\infty) \to 0$

$$\therefore \overline{\tau}_{rz} = -\int_0^\infty g(r,k)\frac{\partial}{\partial r}\left(rJ_1(kr)\right)dr \quad (72)$$

$$J_1(kr) = -J_0(kr) \quad (73)$$

$$\overline{\tau}_{rz} = -k\int_0^\infty g(r,z) r\, J_0(kr)dr \quad (74)$$

$$\int_0^\infty \left((1-\mu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\phi(r,z) r J_0(kr)dr = 0 \qquad \text{for } z = 0 \quad (75)$$

$$\mu\frac{d^2\overline{\phi}}{dz^2} + (1-\mu)k^2\overline{\phi} = 0 \quad \text{for } z = 0 \quad (76)$$

By differentiation,

$$\frac{d\overline{\phi}}{dz} = (c_2(1-kz) - c_1k)\exp(-kz) \quad (77)$$

$$\frac{d^2\overline{\phi}}{dz^2} = (c_2(k^2z - 2k) + c_1k^2)\exp(-kz) \quad (78)$$

$$\frac{d^3\overline{\phi}}{dz^3} = (c_2(-k^3z + 3k^2) - c_1k^3)\exp(-kz) \quad (79)$$

For $z = 0$,

$$\frac{d\overline{\phi}}{dz} = c_2 - c_1k \quad (80)$$

$$\frac{d^2\overline{\phi}}{dz^2} = -2c_2k + c_1k^2 \quad (81)$$

$$\frac{d^3\overline{\phi}}{dz^3} = 3k^2c_2 - k^3c_1 \quad (82)$$

$$\mu(-2c_2k + c_1k^2) + (1-\mu)k^2c_1 = 0 \quad (83)$$

$$-2k\mu c_2 + c_1k^2\mu + c_1k^2 - \mu k^2c_1 = 0 \quad (84)$$

$$-2k\mu c_2 + c_1k^2 = 0 \quad (85)$$

$$c_1k - 2\mu c_2 = 0 \quad (86)$$

$$c_1 = \frac{2\mu c_2}{k} \quad (87)$$

Similarly from Equation (68),

$$(1-\mu)(3k^2c_2 - k^3c_1) - (2-\mu)k^2(c_2 - c_1k) = -\frac{pRJ_1(kR)}{k} \quad (88)$$

$$(1-2\mu)c_2k^2 + c_1k^3 = -\frac{pRJ_1(kR)}{k} \quad (89)$$

$$c_1k + (1-2\mu)c_2 = -\frac{pRJ_1(kR)}{k^3} \quad (90)$$

Solving,

$$\frac{2\mu c_2k}{k} + (1-2\mu)c_2 = -\frac{pRJ_1(kR)}{k^3} \quad (91)$$

$$2\mu c_2 + c_2 - 2\mu c_2 = -\frac{pRJ_1(kR)}{k^3} \quad (92)$$

$$c_2 = -\frac{pRJ_1(kR)}{k^3} \quad (93)$$

$$c_1 = \frac{2\mu}{k}c_2 = \frac{2\mu}{k} \cdot \frac{-pRJ_1(kR)}{k^3} \quad (94)$$

$$c_1 = -\frac{2pR\mu J_1(kR)}{k^4} \quad (95)$$

$$\overline{\Phi} = \left(-\frac{2p\mu R}{k^4}J_1(kR) - \frac{pRJ_1(kR)z}{k^3}\right)\exp(-kz) \quad (96)$$

$$\Phi = -pR\left(\frac{2\mu}{k^4} - \frac{z}{k^3}\right)J_1(kR)\exp(-kz) \quad (97)$$

$$\Phi = -pR\left(zk + 2\mu\right)k^4J_1(kR)\exp(-kz) \quad (98)$$

$$\overline{\sigma}_{zz} = (1-\mu)(c_2(3k^2 - k^3z) - c_1k^3)e^{-kz} - (2-\mu)k^2(c_2(1-kz) - c_1k)e^{-kz} \quad (99)$$

$$\overline{\sigma}_{zz} = -(1+kz)\frac{pRJ_1(kR)}{k}\exp(-kz) \quad (100)$$

By inversion,

$$\sigma_{zz} = -\int_0^\infty k(1+kz)\frac{pRJ_1(kR)}{k}\exp(-kz)J_0(kr)dk \tag{101}$$

$$\sigma_{zz} = -\int_0^\infty (1+kz)pRJ_1(kR)J_0(kr)\exp(-kz)dk \tag{102}$$

$$\sigma_{zz} = -pR\int_0^\infty (1+kz)\exp(-kz)J_1(kR)J_0(kr)dk \tag{103}$$

Integration yields, the vertical stress at any point (r, z) in the semi-infinite linear elastic soil as:

$$\sigma_{zz}(r,z) = p\left\{A - \frac{m}{\pi\sqrt{m^2+(1+t)^2}}\left(\frac{m^2-1+t^2}{m^2+(1-t)^2}E(k) + \frac{(1-t)}{(1+t)}\right)\Pi_0(k,n)\right\} \tag{104}$$

where $E(k)$ is the complete elliptic integral of the second kind of modulus k and parameter, n

$\Pi_0(k,n)$ is the complete elliptic integral of the third kind of modulus k, and parameter, n

$$m = \frac{z}{R} \tag{105}$$

$$t = \frac{r}{R} \tag{106}$$

$$k^2 = \frac{4t}{m^2+(t+1)^2} \tag{107}$$

$$n = \frac{-4t}{(t+1)^2} \tag{108}$$

$$A = \begin{cases} 1 & r < R \\ \tfrac{1}{2} & r = R \\ 0 & r > R \end{cases} \tag{109}$$

In general, the vertical stress at any point (r, z) in the elastic half space due to the uniformly distributed load p, on the circular formulation area is:

$$\sigma_{zz}(r,z) = p(I_{A'} + I_{B'}) \tag{110}$$

where $I_{A'}$ and $I_{B'}$ are functions of the dimensionless parameters z/R and r/R. $I_{A'}$ and $I_{B'}$ are tabulated as shown in Tables 1, 2, 3, and 4 in the Appendix.

On the axis $r = 0$ which is the vertical axis directly under the center of the circular foundation,

$$\sigma_{zz}(r=0,z) = -pR\int_0^\infty (1+kz)e^{-kz}J_1(kR)dk \tag{111}$$

$$\sigma_{zz}(r=0,z) = -pR\int_0^\infty e^{-kz}J_1(kR)dk - pR\int_0^\infty kze^{-kz}kz\,J_1(kR)dk \tag{112}$$

$$\sigma_{zz}(0,z) = -p\left(-1+z^3(z^2+R^2)^{-3/2}\right) \tag{113}$$

$$\sigma_{zz}(0,z) = p\left(1+\frac{z^3}{(z^2+R^2)^{3/2}}\right) \tag{114}$$

$$\sigma_{zz}(0,z) = p\left(1-\frac{(z^2)^{3/2}}{(z^2+R^2)^{3/2}}\right) = p\left(1-\left(\frac{z^2}{z^2+R^2}\right)^{3/2}\right) \tag{115}$$

$$\sigma_{zz}(r=0,z) = p\left(1-\left(\frac{1}{1+(R/z)^2}\right)^{3/2}\right) \tag{116}$$

$$\sigma_{zz}(r=0,z) = p\left(1-\left(1+\left(R/z\right)^2\right)^{-3/2}\right) = pI_c(R/z) \tag{117}$$

This same expression (result) is obtained from the general expression for vertical stress at any point (r, z) in the semi-infinite soil by putting $r = 0$ in the expression Equation (104). Then for $r = 0$, $t = 0$, $n = 0$, $k = 0$ and

$$\sigma_{zz}(r=0,z) = p\left(A - \frac{m}{\pi\sqrt{(1+m^2)}}\frac{m^2-1}{m^2+1}E(0) + \Pi_0(0,0)\right) \quad (118)$$

$$\sigma_{zz}(r=0,z) = p\left(1 - \frac{m^3}{(1+m^2)^{3/2}}\right) = p\left(1 - \frac{(m^2)^{3/2}}{(1+m^2)^{3/2}}\right) \quad (119)$$

$$\sigma_{zz}(0,z) = p\left(1 - \left(\frac{m^2}{1+m^2}\right)^{3/2}\right) = p\left(1 - \left(\frac{1}{1+m^{-2}}\right)^{3/2}\right) \quad (120)$$

$$\sigma_{zz}(r=0,z) = p\left(1 - \left(1 + \left(R/z\right)^2\right)^{-3/2}\right) = pI_c(R/z) = p\left(1 - (1+m^{-2})^{-3/2}\right) \quad (121)$$

$I_c(R/z)$ is the non-dimensional stress influence coefficient called the vertical stress influence coefficient for points at a depth z under the center of a circular foundation area of radius R.

Similarly,

$$\sigma_{rr} = pR\int_0^\infty J_0(kr)J_1(kR)(1-kz)e^{-kz}dk - pR\int_0^\infty \frac{J_1(kr)J_1(kR)}{kr}(1-2\mu-kz)e^{-kz}dk \quad (122)$$

$$\sigma_\theta = pR\int_0^\infty J_0(kr)J_1(kR)2\mu e^{-kz}dk + pR\int_0^\infty \frac{J_1(kr)J_1(kR)}{kr}(1-2\mu-kz)e^{-kz}dk \quad (123)$$

$$\tau_{rz} = -pR\int_0^\infty J_1(kR)J_1(kr)kz\,e^{-kz}dk \quad (124)$$

$$u = \frac{-pR}{2G}\int_0^\infty \frac{J_1(kr)J_1(kR)}{k}(1-2\mu-kz)e^{-kz}dk \quad (125)$$

$$w = \frac{pR}{2G}\int_0^\infty \frac{J_0(kr)J_1(kR)}{k}(2-2\mu-kz)e^{-kz}dk \quad (126)$$

On the axis of the circular foundation where $r = 0$, for any z,

$$J_0(kr)|_{r=0} = J_0(kr=0) = J_0(0) = 1; \text{ and } J_1(0) = 0 \quad (127)$$

$$\sigma_{rr} = pR\int_0^\infty J_1(kR)(1-kz)e^{-kz}dk - pR\int_0^\infty \frac{J_1(0)J_1(kR)}{kr}(1-2\mu-kz)e^{-kz}dk \quad (128)$$

$$\sigma_{rr} = pR\int_0^\infty J_1(kR)(1-kz)e^{-kz}dk = \sigma_{rr}(r=0,z) \quad (129)$$

$$\sigma_{rr}(r=0,z=0) = pR\int_0^\infty J_1(kR)e^0dk = pR\int_0^\infty J_1(kR)dk \quad (130)$$

$$\sigma_{\theta\theta} = pR\int_0^\infty J_0(kr)J_1(kR)2\mu e^{-kz}dk + pR\int_0^\infty \frac{J_1(0)J_1(kR)}{kr}(1-2\mu-kz)e^{-kz}dk \quad (131)$$

$$\tau_{rz}(r=0,z) = pR\int_0^\infty J_1(0)J_1(kR)kz\,e^{-kz} = 0 \quad (132)$$

$$u(r=0,z) = \frac{-pR}{2G}\int_0^\infty \frac{J_1(0)J_1(kR)}{k}(1-2\mu-kz)e^{-kz}dk \quad (133)$$

$$w(r=0,z) = \frac{pR}{2G}\int_0^\infty \frac{J_0(0)J_1(kR)}{k}(2-2\mu+kz)e^{-kz}dk \quad (134)$$

$$w(r=0,z)=\frac{pR}{2G}\int_0^\infty \frac{J_1(kR)}{k}(2-2\mu+kz)e^{-kz}dk \tag{135}$$

At the surface of the soil, $z = 0$, and:

$$w(r=0,z=0)=\frac{pR}{2G}\int_0^\infty \frac{J_1(kR)}{k}(2-2\mu)e^{0}dk \tag{136}$$

$$w(0,0)=\frac{pR(1+\mu)}{E}\int_0^\infty \frac{J_1(kR)}{k}2(1-\mu)dk \tag{137}$$

$$w(0,0)=\frac{2pR(1-\mu^2)}{E}\int_0^\infty \frac{J_1(kR)}{k}dk \tag{138}$$

$$\sigma_{rr}(r=0,z)=\sigma_{rr}(0,z)=\frac{p}{2}\left\{1-2\mu-\frac{(1-2\mu)z}{(R+z^2)^{1/2}}-\frac{Rz^2}{(R^2+z^2)^{3/2}}\right\} \tag{139}$$

$$\tau_{rz}=0=\tau_{rz}(r=0,z) \tag{140}$$

$$w=\frac{2(1-\mu^2)}{E}p\left\{(R^2+z^2)^{1/2}-z\right\}-\left(\frac{1+\mu}{E}\right)pz\left\{1-\frac{z}{(R^2+z^2)^{1/2}}\right\}=w(r=0,z) \tag{141}$$

$$u=0=u(r=0=z) \tag{142}$$

At the surface of the soil, $z = 0$, at the point where $r = 0$, the vertical displacement is:

$$w(r=0,z=0)=w(0,0)=\frac{2(1-\mu^2)}{E}pR \tag{143}$$

Similarly,

$$\sigma_{rr}=\frac{p}{2}(1-2\mu)=\sigma_{rr}(r=0,z=0) \tag{144}$$

V. DISCUSSION

In this study, the Hankel transformation method has been successfully implemented to solve the three dimensional axisymmetric elasticity problem of uniformly loaded circular foundation on semi-infinite soil with $0 \le r \le \infty,\ 0 \le z \le \infty.$ The soil occupying the half space region was assumed homogeneous isotropic and linear elastic. Stress based formulation technique was adopted. Love stress functions were applied to simplify the mathematically rigorous problem of finding a biharmonic stress function of the three dimensional cylindrical spatial coordinates $\phi(r,z)$ that satisfied the boundary conditions on stresses as well as the bounded requirements of the solutions.

The Hankel transformation was applied to the fourth order biharmonic stress compatibility Equation (1) to obtain the much simpler fourth order ordinary differential Equation (42) in terms of the stress function $\overline{\phi}(k,z)$ in the Hankel transform space. Equation (42) was solved using the method of trial functions to obtain the general solution in the Hankel transform space as Eqaution (48). Boundedness requirements for the stresses and displacements were applied to obtain the stress function for bounded solutions as Equation (50) which contains two arbitrary constants that were obtained from the traction and deformation boundary conditions. Hankel transformations were similarly applied to the expressions for stresses to obtain the stresses in the Hankel transform space as Equations (56) and (64). Boundary conditions were enforced to obtain the arbitrary constants as Equations (93) and (95). The completely determined stress function was thus obtained as Equation (98). The stresses are found in the Hankel transform space as Equations (100), (122), (123) and (124). The displacements were similarly found in the Hankel transform space as Equations (125) and (126). The solutions were obtained in the physical domain space variables by inversion of the Hankel transforms of the stresses and the displacements. The vertical stress field was obtained for any point in the semi-infinite soil as Equation (103). Integration yielded the general solution for vertical stress at any point in the elastic half space due to the uniformly distributed load over the circular area as Equation (104). The solution for σ_{zz} was presented in terms of dimensionless influence coefficients as Equation (110) where the influence coefficients are tabulated as shown displayed in the Appendix (Tables 1-5). The vertical stress field was also obtained for points in the soil directly beneath the center of the loaded area as Equation (117). By similar inversion procedures, the stresses and displacements were found in the physical domain space variables for points in the elastic half space under the center of the loaded area as Equations (139-142). The z component of the displacement at $r = 0$, $z = 0$ was found as Equation (143); while the radial stress component was obtained as Equation (144). It was observed that the expressions obtained for the stresses and displacements are identical to the solutions found in the technical literature.

VI. CONCLUSIONS

From the study, the following conclusions can be made:

(i) The Hankel transform method is an effective mathematical technique for solving the axisymmetric three dimensional elasticity problem of a uniformly distributed load on a circular area of linear elastic, isotropic homogeneous soil mass considered semi-infinite in extent (elastic half space; $0 \leq r \leq \infty$, $0 \leq z \leq \infty$)

(ii) The Hankel transformation operation simplifies the governing stress compatibility equation which is a fourth order biharmonic partial differential equation (PDE) in terms of the Love stress function $\phi(r, z)$ to a linear fourth order ordinary differential equation with constant coefficients; which is more amenable to mathematical solutions than the original untransformed biharmonic (PDE) equation.

(iii) The Love stress function formulation of the axisymmetric problem simplified the solution process to obtaining a suitable scalar field of the cylindrical coordinate variables that are solutions of the biharmonic stress compatibility equations, and simultaneously satisfied the boundary conditions and the bounded requirements of the stress fields and displacement fields.

(iv) Solutions obtained for the vertical stress distribution were observed to be symmetrical with respect to the vertical axis of symmetry ($r = 0$) which is the vertical axis under the center of the circular loaded area. This is in line with the symmetrical nature of the problem and the symmetry of the applied load about the vertical axis of symmetry ($r = 0$).

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APPENDIX

Vertical stress field due to uniformly distributed load over a circular area in soil idealised as elastic half space

At any arbitrary point (*r*, *z*) in the elastic soil, the increase in the vertical stress (σ_z) at any point located at a depth *z* at any radial distance *r* from the center of the uniformly loaded area can be given in terms of the non-dimensional coefficients $I_{A'}$ and $I_{B'}$ as:

$$\sigma_z = p(I_{A'} + I_{B'})$$

where $I_{A'}$ and $I_{B'}$ are functions of *z*/*R* and *r*/*R*

Table 1
Vertical stress influence coefficients due to uniformly distributed load over a circular area of radius *R* in semi-infinite linear elastic (elastic half space) soil
Variation of $I_{A'}$ with *z*/*R* and *r*/*R*

z/*R* \ *r*/*R*	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.5**	**2**
0	1.0	1.0	1.0	1.0	1.0	0.5	0	0	0
0.1	0.90050	0.89748	0.88679	0.86126	0.78797	0.43015	0.09645	0.02787	0.00856
0.2	0.80388	0.79824	0.77884	0.73483	0.63014	0.38269	0.15433	0.05251	0.01680
0.3	0.71265	0.70518	0.68316	0.62690	0.52081	0.34375	0.17964	0.07199	0.02440
0.4	0.62861	0.62015	0.59241	0.53767	0.44329	0.31048	0.18709	0.08593	0.03118
0.5	0.55279	0.54403	0.51622	0.46448	0.38390	0.28156	0.18556	0.09499	0.03701
0.6	0.48550	0.47691	0.45078	0.40427	0.33676	0.25588	0.17952	0.10010	
0.7	0.42654	0.41874	0.39491	0.35428	0.29833	0.21727	0.17124	0.10228	0.04558
0.8	0.37531	0.36832	0.34729	0.31243	0.26581	0.21297	0.16206	0.10236	
0.9	0.33104	0.32492	0.30669	0.27707	0.23832	0.19488	0.15253	0.10094	
1	0.29289	0.28763	0.27005	0.24697	0.21468	0.17868	0.14329	0.09849	0.05185
1.2	0.23178	0.22795	0.21662	0.19890	0.17626	0.15101	0.12570	0.09192	0.05260
1.5	0.16795	0.16552	0.15877	0.14804	0.13436	0.11892	0.10296	0.08048	0.05116
2	0.10557	0.10453	0.10140	0.09647	0.09011	0.08269	0.07471	0.06275	0.04496
2.5	0.07152	0.07008	0.06947	0.06698	0.06373	0.05974	0.05555	0.04880	0.03787
3	0.05132	0.05101	0.05022	0.04886	0.04707	0.04487	0.04241	0.03839	0.03150
4	0.02986	0.02976	0.02907	0.02802	0.02832	0.02749	0.02651	0.02490	0.02193
5	0.01942	0.01938				0.01835			0.01573
6	0.01361					0.01307			0.01168
7	0.01005					0.00976			0.00894
8	0.00772					0.00755			0.00703
9	0.00612					0.00600			0.00566
10								0.00477	0.00465

Table 2
Variation of $I_{A'}$ with *z*/*R* and *r*/*R* (*continued*)

z/*R* \ *r*/*R*	**3**	**4**	**5**	**6**	**7**	**8**	**10**	**12**	**14**
0	0	0	0	0	0	0	0	0	0
0.1	0.00211	0.00084	0.00042						
0.2	0.00419	0.00167	0.00063	0.00048	0.00030	0.00020			
0.3	0.00622	0.00250							
0.5	0.01013	0.00407	0.00209	0.00118	0.00071	0.00053	0.00025	0.00014	0.00009
1	0.01742	0.00761	0.00393	0.00236	0.00143	0.00097	0.00050	0.00029	0.00018
1.2	0.01935	0.00871	0.00459	0.00269	0.00171	0.00115			
1.5	0.02142	0.01013	0.00548	0.00325	0.00210	0.00141	0.00073	0.00043	0.00027
2	0.02221	0.01160	0.00659	0.00399	0.00264	0.00180	0.00094	0.00056	0.00036
2.5	0.02143	0.01221	0.00732	0.00463	0.00308	0.00214	0.00115	0.00068	0.00043
3	0.01980	0.01220	0.00770	0.00505	0.00346	0.00242	0.00132	0.00079	0.00051
4	0.01592	0.01109	0.00768	0.00536	0.00354	0.00282	0.00160	0.00099	0.00065
5	0.01249	0.00949	0.00708	0.00527	0.00394	0.00298	0.00179	0.00113	0.00075
6	0.00983	0.00795	0.00628	0.00492	0.00384	0.00299	0.00188	0.00124	0.00084
7	0.00784	0.00661	0.00548	0.00443	0.00360	0.00291	0.00193	0.00130	0.00091
8	0.00635	0.00554	0.00472	0.00398	0.00332	0.00276	0.00189	0.00134	0.00094
9	0.00520	0.00466	0.00409	0.00353	0.00301	0.00256	0.00184	0.00133	0.00096
10	0.00438	0.00397	0.00352	0.00326	0.00273	0.00241			

Table 3
Variation of $I_{B'}$ with z/R and r/R

z/R \ r/R	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.5**	**2**
0	0	0	0	0	0	0	0	0	0
0.1	0.09852	0.10140	0.11138	0.13424	0.18796	0.05388	–0.07899	–0.02672	–0.00845
0.2	0.18857	0.19306	0.20272	0.23524	0.25983	0.08513	–0.07759	–0.04448	–0.01593
0.3	0.26362	0.26787	0.28018	0.29483	0.27257	0.10757	–0.04316	–0.04999	–0.02166
0.4	0.32016	0.32259	0.32748	0.32273	0.26925	0.12404	–0.00766	–0.04535	–0.02522
0.5	0.35777	0.35752	0.35323	0.33106	0.26236	0.13591	0.02165	–0.03455	–0.02651
0.6	0.37831	0.37531	0.36308	0.32822	0.25411	0.14440	0.04457	–0.02101	
0.7	0.38487	0.37962	0.36972	0.31029	0.24638	0.14986	0.06209	–0.00702	–0.02329
0.8	0.38091	0.37408	0.35133	0.30699	0.23779	0.15292	0.07530	0.00614	
0.9	0.36962	0.36275	0.33734	0.29299	0.22891	0.15404	0.08507	0.01795	
1	0.35355	0.34553	0.32075	0.27819	0.21978	0.15355	0.09210	0.02814	–0.01005
1.2	0.31485	0.30730	0.28481	0.24836	0.20113	0.14915	0.10002	0.04378	0.00023
1.5	0.25602	0.25025	0.23338	0.20694	0.17368	0.13732	0.10193	0.05745	0.01385
2	0.17889	0.18144	0.16644	0.15198	0.13375	0.11331	0.09254	0.06371	0.02836
2.5	0.12807	0.12633	0.12126	0.11327	0.10298	0.09130	0.07869	0.06022	0.03429
3	0.09487	0.09394	0.09099	0.08635	0.08033	0.07325	0.06551	0.05354	0.03511
4	0.05707	0.05666	0.05562	0.05383	0.05145	0.04773	0.04532	0.03995	0.03066
5	0.03772	0.03760				0.03384			0.02474
6	0.02666					0.02468			0.01968
7	0.01980					0.01868			0.01577
8	0.01526					0.01459			0.01279
9	0.01212					0.1170			0.01054
10								0.00924	0.00879

Table 4
Variation of $I_{B'}$ with z/R and r/R (*continued*)

z/R \ r/R	**3**	**4**	**5**	**6**	**7**	**8**	**10**	**12**	**14**
0	0	0	0	0	0	0	0	0	0
0.1	–0.00210	–0.00084	–0.00042						
0.2	–0.00412	–0.00166	–0.00083	–0.00024	–0.00015	–0.00010			
0.3	–0.00599	–0.00245							
0.5	–0.00991	–0.00388	–0.00199	–0.00116	–0.00073	–0.00049	–0.00025	–0.00014	–0.00009
1	–0.01115	–0.00608	–0.00344	–0.00210	–0.00135	–0.00092	–0.00048	–0.00028	–0.00018
1.2	–0.00995	–0.00632	–0.00378	–0.00236	–0.00156	–0.00107			
1.5	–0.00669	–0.00600	–0.00401	–0.00265	–0.00181	–0.00126	–0.00068	–0.00040	–0.00026
2	0.00028	–0.00410	–0.00371	–0.00278	–0.00202	–0.00148	–0.00084	–0.00050	–0.00033
2.5	0.00661	–0.00130	–0.00271	–0.00250	–0.00201	–0.00156	–0.00094	–0.00059	–0.00039
3	0.01112	0.00157	–0.00134	–0.00192	–0.00179	–0.00151	–0.00099	–0.00065	–0.00046
4	0.01515	0.00595	0.00155	–0.00029	–0.00094	–0.00109	–0.00094	–0.00068	–0.00050
5	0.01522	0.00810	0.00371	0.00132	0.00013	–0.00043	–0.00070	–0.00061	–0.00049
6	0.01380	0.00867	0.00496	0.00254	0.00110	0.00028	–0.00037	–0.00047	–0.00045
7	0.01204	0.00842	0.00547	0.00332	0.00185	0.00093	–0.00002	–0.00029	–0.00037
8	0.01034	0.00779	0.00554	0.00372	0.00236	0.00141	0.00035	–0.00008	–0.00025
9	0.00888	0.00705	0.00533	0.00386	0.00265	0.00178	0.00066	0.00012	–0.00012
10	0.00764	0.00631	0.00501	0.00382	0.00281	0.00199			

Table 5

Vertical normal stress distribution influence coefficient $I_c(R/z)$ under the center of uniformly loaded circular foundation area of radius *R*.

$$I_c\left(R/z\right) = 1 - \left(1 + \left(R/z\right)^2\right)^{-3/2}$$

R/z	I_c	R/z	I_c	R/z	I_c	R/z	I_c
0.00	0.00000	0.30	0.12126	0.60	0.36949	0.90	0.58934
0.01	0.00015	0.31	0.12859	0.61	0.37781	0.91	0.59542
0.02	0.00060	0.32	0.13605	0.62	0.38609	0.92	0.60142
0.03	0.00135	0.33	0.14363	0.63	0.39431	0.93	0.60734
0.04	0.00240	0.34	0.15133	0.64	0.40247	0.94	0.61317
0.05	0.00374	0.35	0.15915	0.65	0.41058	0.95	0.61892
0.06	0.00538	0.36	0.16706	0.66	0.41863	0.96	0.62459
0.07	0.00731	0.37	0.17507	0.67	0.42662	0.97	0.63018
0.08	0.00952	0.38	0.18317	0.68	0.43454	0.98	0.63568
0.09	0.01203	0.39	0.19134	0.69	0.44240	0.99	0.64110
0.10	0.01481	0.40	0.19959	0.70	0.45018	1.00	0.64645
0.11	0.01788	0.41	0.20790	0.71	0.45789	1.01	0.65171
0.12	0.02122	0.42	0.21627	0.72	0.46553	1.02	0.65690
0.13	0.02483	0.43	0.22469	0.73	0.47310	1.03	0.66200
0.14	0.02870	0.44	0.23315	0.74	0.48059	1.04	0.66703
0.15	0.03283	0.45	0.24165	0.75	0.48800	1.05	0.67198
0.16	0.03721	0.46	0.25017	0.76	0.49533	1.06	0.67686
0.17	0.04184	0.47	0.25872	0.77	0.50259	1.07	0.68166
0.18	0.04670	0.48	0.26729	0.78	0.50976	1.08	0.68639
0.19	0.05181	0.49	0.27587	0.79	0.51685	1.09	0.69104
0.20	0.05713	0.50	0.28446	0.80	0.52386	1.10	0.69562
0.21	0.06268	0.51	0.29304	0.81	0.53079	1.11	0.70013
0.22	0.06844	0.52	0.30162	0.82	0.53763	1.12	0.70457
0.23	0.07441	0.53	0.31019	0.83	0.54439	1.13	0.70894
0.24	0.08057	0.54	0.31875	0.84	0.55106	1.14	0.71342
0.25	0.08692	0.55	0.32728	0.85	0.55766	1.15	0.71747
0.26	0.09346	0.56	0.33579	0.86	0.56416	1.16	0.72163
0.27	0.10017	0.57	0.34427	0.87	0.57058	1.17	0.72573
0.28	0.10704	0.58	0.35272	0.88	0.57692	1.18	0.72976
0.29	0.11408	0.59	0.36112	0.89	0.58317	1.19	0.73373

Table 5: *Continued*

R/z	I_c	R/z	I_c	R/z	I_c	R/z	I_c
1.20	0.73763	1.60	0.85112	2.00	0.91056	5.00	0.99246
1.21	0.74147	1.61	0.85312	2.02	0.91267	5.20	0.99327
1.22	0.74525	1.62	0.85507	2.04	0.91472	5.40	0.99396
1.23	0.74896	1.63	0.85700	2.06	0.91672	5.60	0.99457
1.24	0.75262	1.64	0.85890	2.08	0.91865	5.80	0.99510
1.25	0.75622	1.65	0.86077	2.10	0.92053		
1.26	0.75976	1.66	0.86260	2.15	0.92499	6.00	0.99556
1.27	0.76324	1.67	0.86441	2.20	0.92914	6.50	0.99648
1.28	0.76666	1.68	0.86619	2.25	0.93301		
1.29	0.77003	1.69	0.86794	2.30 2.35	0.93661 0.93997		
1.30	0.77334	1.70	0.86966	2.40	0.94310	7.00	0.99717
1.31	0.77660	1.71	0.87136	2.45	0.94603	7.50	0.99769
1.32	0.77981	1.72	0.87302	2.50	0.94877		
1.33	0.78296	1.73	0.87467	2.55	0.95134	8.00	0.99809
1.34	0.78606	1.74	0.87628	2.60	0.95374		
1.35	0.78911	1.75	0.87787	2.65	0.95599	9.00	0.99865
1.36	0.79211	1.76	0.87944	2.70	0.95810		
1.37	0.79507	1.77	0.88098	2.75	0.96009	10.00	0.99901
1.38	0.79797	1.78	0.88250	2.80	0.96195		
1.39	0.80083	1.79	0.88399	2.85	0.96371	12.00	0.99943
1.40	0.80364	1.80	0.88546	2.90	0.96536	14.00	0.99964
1.41	0.80640	1.81	0.88691	2.95	0.96691		
1.42	0.80912	1.82	0.88833			16.00	0.99976
1.43	0.81179	1.83	0.88974	3.00	0.96838		
1.44	0.81442	1.84	0.89112	3.10	0.97106	18.00	0.99983
1.45	0.81701	1.85	0.89248	3.20	0.97346		
1.46	0.81955	1.86	0.89382	3.30	0.97561	20.00	0.99988
1.47	0.82206	1.87	0.89514	3.40	0.97753		
1.48	0.82452	1.88	0.89643	3.50	0.97927	25.00	0.99994
1.49	0.82694	1.89	0.89771	3.60	0.98083	30.00	0.99996
1.50	0.82932	1.90	0.89897	3.70	0.98224		
1.51	0.83167	1.91	0.90021	3.80	0.98352	40.00	0.99998
1.52	0.83397	1.92	0.90143	3.90	0.98468		
1.53	0.83624	1.93	0.90263			50.00	0.99999
1.54	0.83847	1.94	0.90382	4.00	0.98573		
1.55	0.84067	1.95	0.90498	4.20	0.98757	100.00	1.00000
1.56	0.84283	1.96	0.90613	4.40	0.98911		
1.57	0.84495	1.97	0.90726	4.60	0.99041	∞	1.00000
1.58	0.84704	1.98	0.90838	4.80	0.99152		
1.59	0.84910	1.99	0.90948				

AUTHOR PROFILE

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