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# Multifractal Space – Time Dynamic of Tropical Rainfall

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***Abstract*—For several years, hydrology has required a holistic theory of space-time evolution of mesoscale tropical rainfall which would allow us to take advantage from it to its application in water resources engineering. Nevertheless, various efforts have covered many techniques from the deterministic to stochastic models but not one gives a complete and satisfactory answer to how rainfall behaves in all its space-time scales and even more how its emerging patterns change. This article reviews some facts that show multifractality is an intrinsic property of rainfall and a coupled theory between multifractal theory and stochastic processes could lead us to a better understanding of tropical rainfall and its forecast.**

**Keywords -** Multifractal, Stochastic processes, Rainfall, Space - Time Dynamic,

## I. INTRODUCTION

According to the scientific literature there are some prevalence limitations in the hydrologic modelling related to questions of scale, nonlinearity and uniqueness of place. The mentioned above suggests looking through alternative tools to describe physical (or geophysical) processes, one of them based on the geometrical composition of observed data [13, 21, 25]. This geometrical composition has allowed us to recognize that rainfall, as one of the highly complex geophysical processes, exhibits a multifractal structure in its description [12].

Multifractal measures are related to the study of physical distribution (or any other amounts) on a geometrical support. The general idea of a multifractal can be understood as a geometric object in terms of intertwined fractal subsets which have different scaling exponents [2]. The multifractal analysis has been extended in its applications to provide a technique for analysing complex systems, where its applications has solved some problems in the study of turbulence phenomena [3].

The identification of multifractal patterns among rainfall observed data records suggests the estimation of nonlinear statistical attributes designed for this purpose, such as multifractal spectra do. To understand what multifractal spectra is and how it works, one can consider a population generated by a one-dimensional binomial multiplicative process formed by $N$ members distributed on a line segment $S = [0, 1]$ (see Fig. 1).

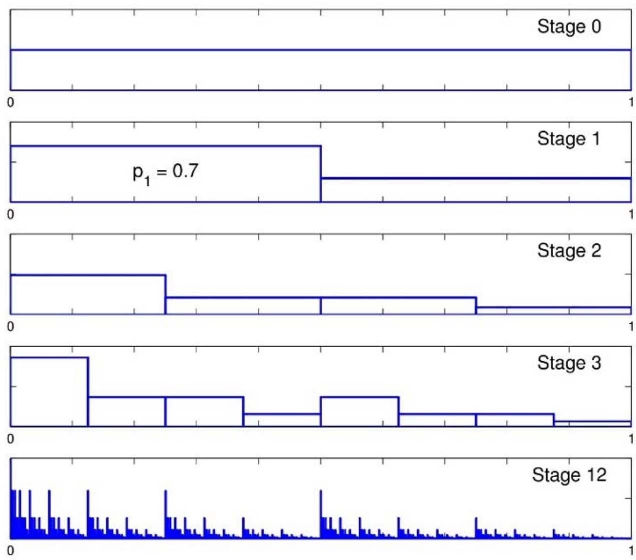


Fig. 1. Construction of a binomial multiplicative process with parameter $p_1 = 0{,}70$ and the measure is defined on a one-dimensional Euclidean support. This binomial multiplicative process is plotted for the first 12 stages.

In order to characterize this distribution is divided the segments into pieces (or cells) of length $\delta = 2^{-n}$, such as, $N = 2^n$ cells are necessary to cover the segment S. Here, n is the number of generations (or stage) in the binary subdivision of line segments and the cells are labelled by the indexes $i = 0, 1, 2, 3, \dots, N-1$. The distribution of the population over the line is specified, in the resolution $\delta$, by the number $N_i$ of members of the population in the $i$-th cell. The fraction of the total population $\mu_i = N_i/N$ is a convenient measure for the content of the cell and is certain if the set $M$, given by $M = \sum \mu_i$, is satisfied.

For a binomial multiplicative process in the $n$-th generation (as exhibited in Fig. 1), $N(\alpha)$ line segments of length $\delta = 2^{-n}$ have the same measure $\mu(\alpha)$. These segments form a subset $S(\alpha)$ over the geometric support $S = [0, 1]$ and this set defined by $N(\alpha)$ segments has a singularity with Lipschitz-Hölder exponent $\alpha$ and a fractal dimension $f(\alpha)$ (See reference [2] for a further explanation of the singularity exponent estimation).

The most important here is to see how $f(\alpha)$ behaves for every singularity $\alpha$. As it is exhibited in Fig. 2 a parabolic function which represents how a pattern, coming from a binomial multiplicative process, can describe several scales and fractal dimensions in order to represent what is called in geophysics as multifractality [2, 3, 21]. For instance, the case that is illustrated in Fig. 2 shows a representation of a multifractal spectrum for the observed dissipation field of fully developed turbulence [11].

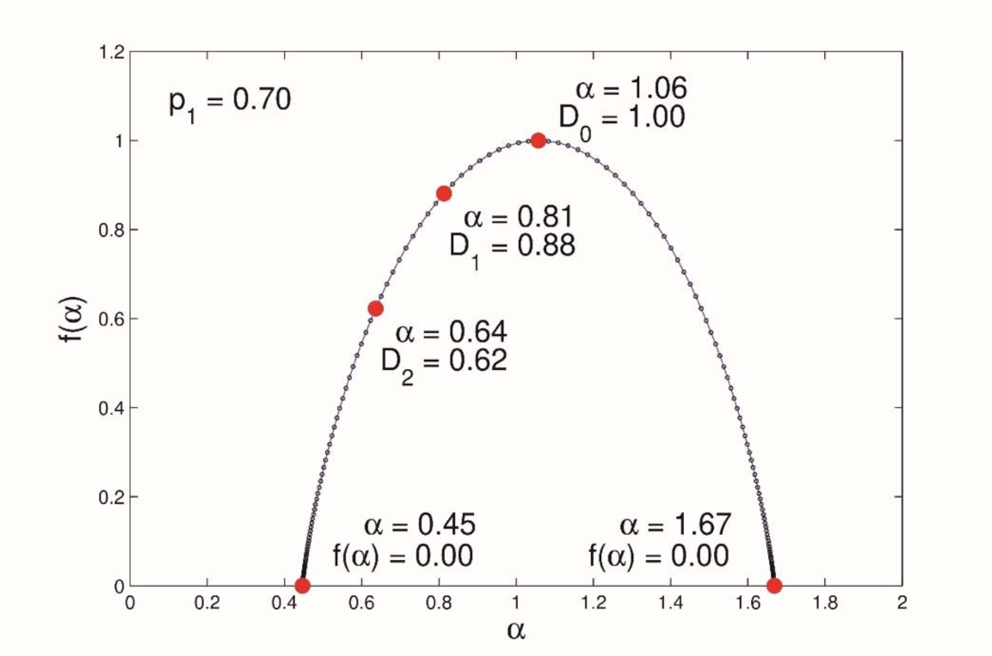


Fig. 2 Fractal dimension of subsets $S(\alpha)$ as a function of α to a binomial multiplicative process with $p_1 = 0{,}70$.

In similarly way to the turbulence processes which exhibit a well-defined multifractal spectrum, other geophysical processes show an akin behaviour. The multifractal spectra of Fig. 3 and Fig. 4 represents the structure of punctual rainfall observations located at the tropical region. The illustrations were taken out from the work developed in [19]. The rainfall data records were gotten from 21 gauges which are distributed over the metropolitan area of Bogotá and its periphery. These rain gauges belong to the Drinking and Sewer Water Management Company of Bogotá (EAAB E.S.P) and the rainfall data were recorded by rainfall gauges at the resolution of 30 minutes during the period defined by the years 1995 to 1999.

The multifractal spectrums exhibited in Fig. 3 and Fig. 4 show an ensemble of curves, moderately symmetric that identify heterogeneous scales in rainfall data records and a complex group of singularities associated to a thorny pattern. In the same way, these multifractal spectra characterize rainfall as a non-linear physical process and allow us to think that its similarity to the binomial multiplicative process, can be geometrically explained by an akin procedure such as Meneveau and Sreenivasan in [11] did for the turbulent dissipation, another non-linear process.

As a complementary illustration of multifractal spectrum in rainfall, is suggested to see the work [4], which applied several methodologies to build the multifractal spectrum to 47 rain gauges located in the tropical Andes of Colombia. It is worth to express that multifractal analysis offers not only statistical quantities, but also to evaluate geometric features that indicate the shape and distribution of records in the space – time domain. Since the potentiality of the multifractal spectrum, this tool has been strongly suggested to be applied on hydrological sciences [4, 12, 14, 19, 20, 21].

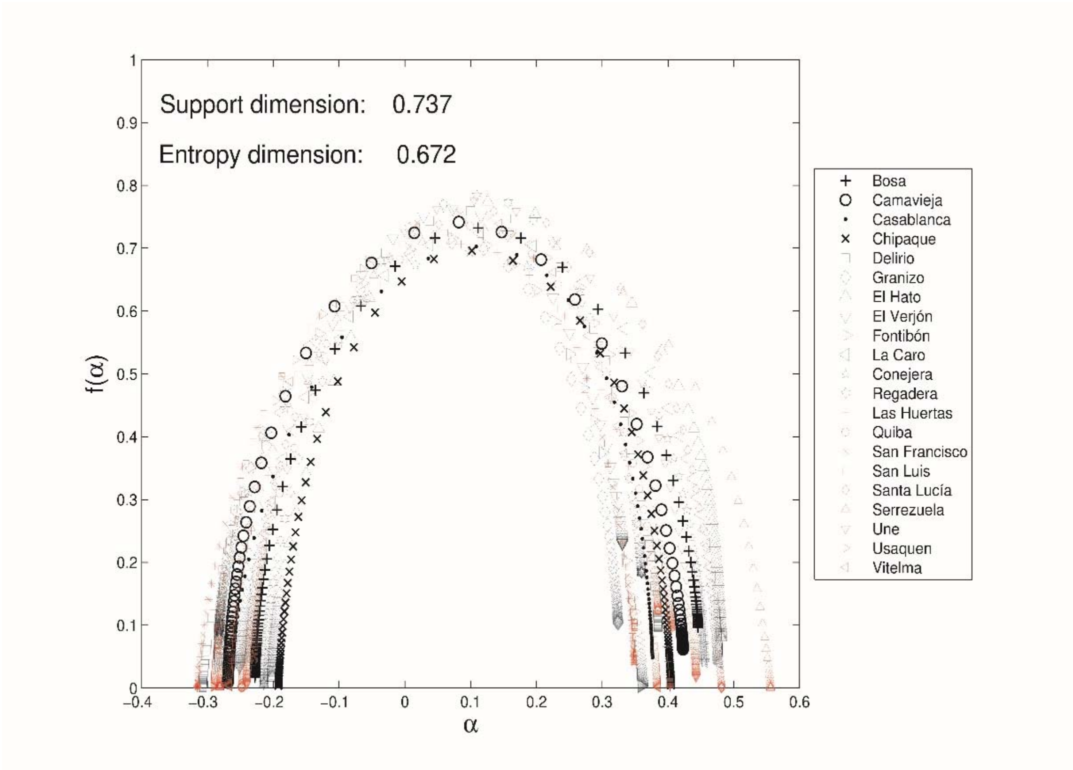


Fig. 3 The multifractal spectrum of 21 rainfall time series belonging to the measurement of 21 rain gauges distributed over the metropolitan area of Bogotá during the period 1995– 1999.

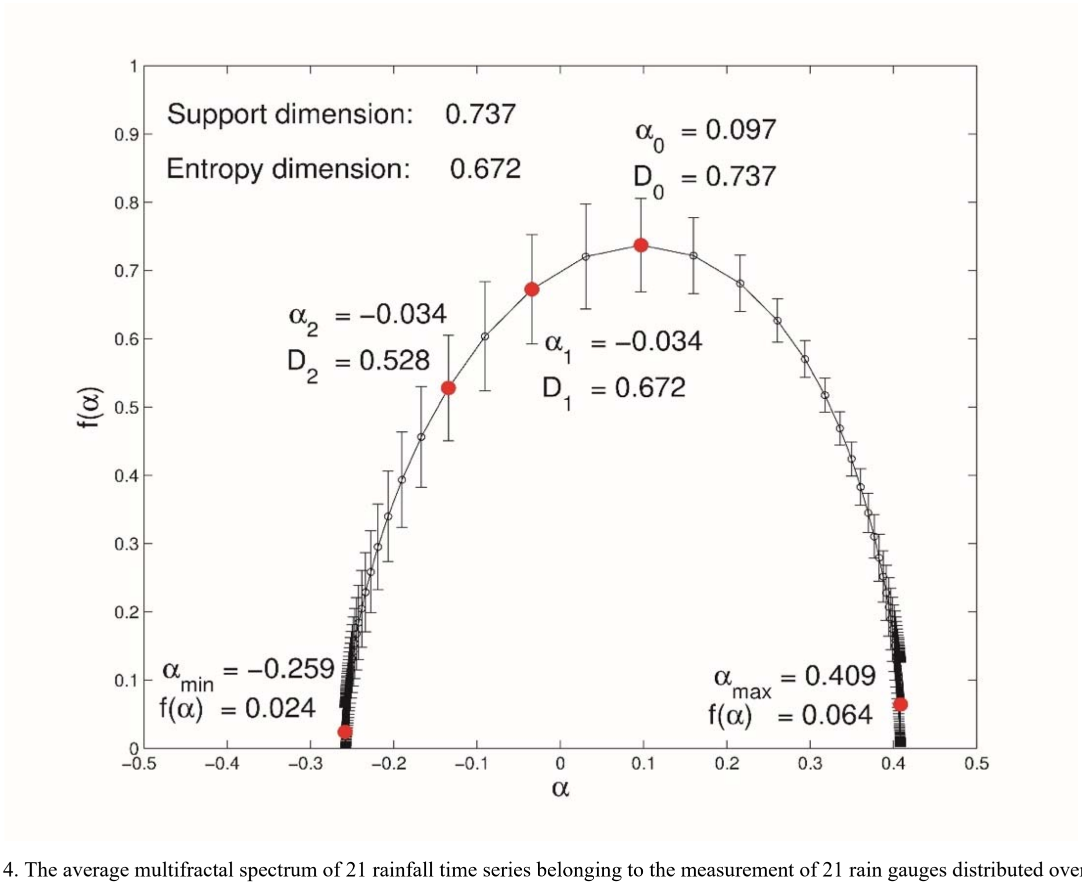


Fig. 4. The average multifractal spectrum of 21 rainfall time series belonging to the measurement of 21 rain gauges distributed over the metropolitan area of Bogotá during the period 1995–1999.

## II. LOOKING FOR A MODEL OF RAINFALL

The starting point of rainfall modelling came from long-time ago. One of the most important scientific reviews is presented by Waymire and Gupta [27,28,29]. In a collection of three papers, Waymire and Gupta [27,28,29] show the historic efforts to build a rainfall model from a stochastic point of view, besides, they clarify the relevant statistical aspects of rainfall that it has been looked for a suitable representation. From the temporal structure of rainfall, it has always been necessary to compute the number of rainfall events, to know the rainfall amount, its distribution in the time domain and so on. Waymire and Gupta [27,28,29] highlight that the number of rainfall events is associated to the thermodynamic instability of the atmosphere and it seems to exhibit a clustering dependence which is difficult to represent by stochastic models.

It is clear that rainfall patterns exhibit highly complex geometries and their data have been so hard to model by stochastic models or even by coupled models between physically based representations and stochastic processes. Although significant successes have been attained using these methods, they possess an important limitation i.e. they are constructed to preserve only some relevant statistical attributes. Since the last reason, the glance of rainfall modelling has changed fairly. Nowadays, there exists important efforts in the applications of derived multifractal models [1, 7, 14, 19, 21, 23, 24, 26] and even more in multifractal random cascades models [ 5, 6, 8, 9, 15, 16, 17, 30].

A remarkable feature of some multifractal models as that introduced by Puente and Obregón [23, 24] under the name as fractal – multifractal (FMF) approach is its entirely deterministic mathematical background, furthermore this model does not require any statistical assumptions (e.g. stationarity, ergodicity and a minimal record length) to model non-linear patterns such as rainfall is [14, 19, 23, 24,]. The FMF approach is obtained by the projection of the graph of a fractal interpolation function illuminated by a multifractal measure (see Fig. 5). These functions interpolate a given set of $N+1$ ordered points along the plane. Mathematically, the fractal-multifractal model is represented by the graph $G = \{(x, f(x)) | x \in [0,1]\}$ of a fractal interpolation function $f: x \to y$ passing by the points: $\{(x_n, y_n) | x_0 < \cdots < x_n,\ n = 0, 1, \ldots., N\}$, which are defined by the unique attractor of N affine maps:

$$W_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}, \quad n = 1, \ldots, N \qquad (1)$$

Where the vertical scaling parameter $d_n$ satisfy $|d_n| < 1$ and the other parameters $a_n$, $c_n$, $e_n$ and $f_n$ are based on the contracting initial conditions:

$$W_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \qquad (2)$$

$$W_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad (3)$$

Pioneers researchers applied the FMF approach to describe and to represent high-resolution rainfall time series [14, 23, 24]. Their results exhibit an outstanding fit between the observed and simulated patterns. Such results show that a geometric methodology can get the timing and size of the largest pulses and the noisiest fluctuations in order to get an overall appearance of rainfall data sets. In spite of these remarkable results, the extension of the FMF approach to describe the space-time evolution could be limited since the loss of its parsimony (i.e. the increasing in the number of model parameters), and even it would also complicate the inverse problem of it [12].

On the other hand, the multifractal random cascade models try to describe the multifractal structure of rainfall, suggesting that its structure is consistent with a multiplicative cascade mechanism. [5, 15, 16]. The advantage of random cascade models is that it provides a geometrical-statistical framework to understand the intermittency and variability of a rainfall field pattern over a wide range of space-time scales [5]. The mathematical background begins with a given mass density, say $W_0$, distributed uniformly over some bounded physical region (e.g. rainfall field) $J = [0,1]^d$. If $J$ is a unit square $(d = 2)$, $J$ is subdivided into $b = N^2$ sub squares of side length $1/N$. Let $J(\sigma), \sigma = 1, 2, 3, \ldots, b = N^2$ denote the partition of $J$ into these sub-squares. Now the mass density $W_0$ is distributed over each of these $b$ sub squares as $W_0 W_1(1), W_0 W_1(2), \ldots, W_0 W_1(b)$, respectively, where $W$'s are mutually independent random variables with identical probability distribution (iid). In the case of canonical cascades, it is stipulated that $E[W] = 1$, which means that the ensemble average of the mass density $W_0$ is conserved after this redistribution. The last procedure is carried on iteratively such that at the $n$-th generation number (or stage), the unit square $J$ is divided into $N^{2n} = b^n$ sub squares of side length $1/N^n$.

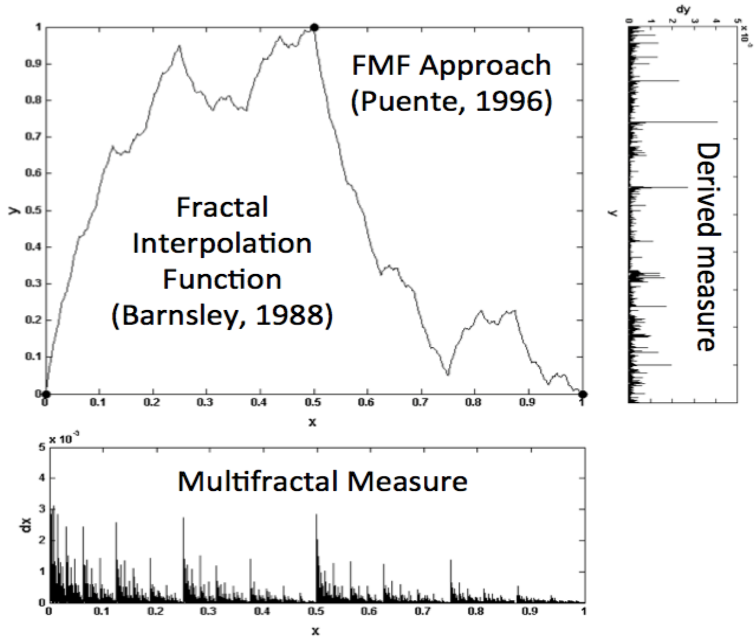


Fig. 5. Sketch of the fractal - multifractal approach. The central panel shows a fractal interpolation function of three points at $\{0,0\}$, $\{\frac{1}{2}, 1\}$, and $\{1,0\}$. At the bottom panel is exhibited the multifractal measure projected from fractal interpolation function and at top right panel is exhibited the derived measure (or model output) which is another projection of the fractal interpolation function.

It is highlighted that random cascade models are more suitable to formulate a space-time evolution model. The most important works related to this kind of models began with Thomas Over [15, 16, 17] and David Marzan et al. [10], who proposed space-time rainfall models based on the multifractal random cascade theory. Both models require only a few parameters to model the observed intermittency and hierarchy of scales in the space-time domains. The theory proposed by Over [15] suggest that the rainfall fields are constructed from discrete multiplicative cascades of independent and identically distributed random variables. Furthermore, the time dimension of the process has an evolutionary behaviour that distinguishes the past, the present and the future, while the spatial dimension have an isotropic stochastic structure. On the other hand, Marzan et al. [10] proposed a space-time model based on scaling dynamics and introduced an important fact to couple the physical processes associated to rainfall: the scaling symmetries in space and time arising from the Navier-Stokes equations at large Reynolds numbers (characteristic of the atmospheric turbulence), should lead to a similar scaling behaviour for active scalar fields.

## III. Thermodynamic Linkages in Rainfall

As it was mentioned before, Marzan et al. [10] introduced an important fact in the modelling of rainfall processes: to introduce a physical mechanism involved in the development of rainfall. Similarly, Perica and Foufoula-Georgiou [22] began to formulate a predictive relationship between statistical characteristic of rainfall and meteorological parameters of the storms. This research would want to find an empirical connection between the scaling parameters (for standardized rainfall fluctuation) and thermodynamic indices of the early storm environment.

The first step in the Perica and Foufoula-Georgiou [22] research was digging the dependence of scaling parameters on the storm type, which resulted in the unexpected evidence of non-relationship in the range of scales of 8 – 64 km. Nevertheless, the scaling parameters showed to be more dependent on the intensity of convective instability of the pre-storm environment. The second step was to validate the hypothesis that since the scale invariance on the developed parameterization, this last might be connected to physical characteristics of storms. To carry on this validation, they used the data from a dense network of rawinsonde stations to characterize the storm environment and determine some thermodynamic parameters such as CAPE (i.e. Convective Available Potential Energy), CIN (i.e. Convective Inhibition), LFC (i.e. Level of Free Convection) and so much more. Perica and Foufoula-Georgiou [22] developed some regression analysis to qualify the relation between the environmental and scaling parameters of spatial standardized rainfall fluctuations, finding a strong correlation between the scaling exponent $H_i$ and the CAPE. This relation imply that a single thermodynamic parameter is capable to explain, approximately, 60% of the variance of the scaling parameter $H_i$, moreover this relation was able to be account for based on physical and statistical arguments, i.e., high CAPE values coincide with high rainfall intensities and high rainfall intensities are more likely to occur in the neighbourhood of pixels with high intensity, which implies relatively small fluctuation of the fields.

Another outstanding result related to the linkage of thermodynamic parameters and the statistical multiscale structure of rainfall, is reported by Parodi et al. [18]. They adopt raindrop terminal velocity as a physical parameter to explain the convective rainfall over a wide range of scales. This contribution summed to the results gotten by Perica and Foufoula-Georgiou [22] opened the door to predict the evolutionary statistical/scaling structure of rainfall fields via relationship between thermodynamic (or microphysical) parameters and the statistical description of rainfall. Future works must be leaded to find theoretical evidence of this relationship.

## IV. Final Remarks

Following to Lovejoy and Schertzer [8] ideas, to establish a non-linear theory that allow to explain the atmospheric dynamic over a wide range of scales, is imperative to solve. In addition, some scientific issues should require for answer: 1) there is not an theoretical relationship between phenomenological multifractal cascade models and the underlying dynamical equations of atmosphere, 2) it is not clear which is the physical nature of cascade fluxes and which one is relevant to understand the climate dynamic, 3) there is not valid space-time scaling function for all atmospheric fields, and 4) there is not a known limit of the cascade theory since there is observational evidence that the cascade structure extends to planetary scales and large time scales.

According to the arguments presented here, the intermittences and fluctuations detected on rainfall patterns keep multifractal properties. These multifractal properties are intrinsic associated to the distribution and the scaling of the surrogate parameters involved in the field, nevertheless, it is necessary to get a better comprehension about the mechanism that explain the interrelationship between physical processes into the space-time evolution of rainfall fields. It is necessary to formulate the foundation of an integrated theory toward the holistic understanding of climate and weather dynamic, whose results would contribute to some application in engineering and the management of water resources.

## References

[1] A. Cortis, C. Puente and B. Sivakumar. "Encoding hydrologic information via a fractal geometric approach and its extensions", Journal of Stochastic Environmental Research and Risk Assessment, 2009.
[2] J. Feder, Fractals, New York, USA: Plenum Press, 1988.
[3] U. Frish, and G. Parisi, "On the singularity structure of fully developed turbulence", Turbulence and Predictability in Geophysical Fluid Dynamic and Climate Dynamic, 1985.
[4] J. Gómez and G. Poveda, "Estimación del espectro multifractal para series de precipitación horaria en los Andes Tropicales de Colombia", Revista Academia Colombiana de Ciencias Exactas, Físicas y Naturales, vol. 32:125, pp. 483-502, 2008.
[5] V. Gupta and E. Waymire "A statistical analysis of mesoscale rainfall as a random cascade", Journal of American Meteorological Society, vol. 32, pp. 251-267, 1993.
[6] J. Hernández, "Desagregación especial y espacio – temporal de la lluvia tropical y su respuesta hidrológica: implicaciones ante el cambio climático", M. Eng. Thesis, Universidad Nacional de Colombia, Medellín, Colombia, 2008.
[7] H. Huang, C. Puente and A. Cortis, "Geometric harnessing of precipitation records: re-examining four storms from Iowa City", Journal of Stochastic Environmental Research and Risk Assessment, 2012.
[8] S. Lovejoy and D. Schertzer, "Towards a new synthesis for atmospheric dynamics: space-time cascade", Journal of Atmospheric Research, vol. 96, pp. 1-52, 2010.
[9] S. Lovejoy and D. Schertzer, The Weather and Climate: Emergent Laws and Multifractal Cascades, New York, USA: Cambridge, 2013.
[10] D. Marzan, D. Schertzer and S. Lovejoy, "Casual space-time multifractal processes: predictability and forecasting of rain fields", Journal of Geophysical Research, vol. 101, D21: 26, pp. 333-346, 1996.
[11] C. Meneveau and K. Sreenivisan, "Simple multifractal cascade model for fully developed turbulence", Physical Review Letters, vol. 59, pp. 1424-1427, 1987.
[12] O. Mesa and V. Peñaranda, "Complejidad de la estructura espacio – temporal de la precipitación", Revista Academia Colombiana de Ciencias Exactas, Físicas y Naturales, vol. 39:152, pp. 304-320, 2015.
[13] E. Monroy, J. González, and V. Peñaranda. Introducción a la formulación de planes de manejo y protección de acuíferos. Tunja, Colombia: Universidad Santo Tomás, 2010.
[14] N. Obregón "A New Geometry Approach to Study One-Dimensional Complex Data Set", Doctoral Dissertation, Ph.D. in Hydrologic Sciences, University of California Davis, USA, 1998.
[15] T. Over, "Modelling space-time mesoscale rainfall using random cascades", Doctoral Dissertation, Ph.D. in Geophysics, University of Colorado, USA, 1995.
[16] T. Over and V. Gupta, "Statistical analysis of mesoscale rainfall: dependence of a random cascade generator on large-scale forcing", Journal of Hydrology, vol. 33, pp. 1526-1542, 1994.
[17] T. Over and V. Gupta, "A space-time theory of mesoscale rainfall using random cascades", Journal of Geophysical Research, vol. 101:26, pp. 319-331, 1996.
[18] A. Parodi, E. Foufoula-Georgiou and K. Emanuel, "Signature of microphysics on spatial rainfall statistics", Journal of Geophysical Research, vol. 116, D14119, doi:10.1029/2010JD015124, 2011.
[19] V. Peñaranda, "Representación geométrica determinística de registros de precipitación puntual en Bogotá D.C. con el modelo Fractal-Multifractal", M. Eng. Thesis, Universidad Nacional de Colombia, Bogotá, Colombia, 2008.
[20] V. Peñaranda, "Rainfall Complex Pattern Trend Via Multifractal Analysis", Hydrology and Ecology: Ecosystems, Groundwater and Surface Water – HydroEco'2011, 2011, Abstract 336, Vienna, Austria.
[21] V. Peñaranda, E. Monroy, and I. Rousta. "A geometric approach for the modeling of heterogeneity in physical structures and processes in water sources". International Journal of Engineering and Technology (IJET). vol. 10, pp. 1806-1812, 2018.
[22] S. Perica and E. Foufoula-Georgiou, "Linkage of Scaling and Thermodynamic Parameters of Rainfall. Results from Midlatitude Mesoscale Convective Systems", Journal of Geophysical Research, vol. 101: D3, pp. 7431-7448, 1996.
[23] C. Puente, "A new approach to hydrologic modelling: derived distribution revisited", Journal of Hydrology, vol. 187, pp. 65-80, 1996.
[24] C. Puente and N. Obregón, "A deterministic geometric representation of temporal rainfall: results for a storm in Boston", Journal of Water Resources Research, vol. 32, pp. 2825-2839, 1996.

[25] I. Rousta, F. Javadizadeh, F. Dargahian, H. Ólafsson, A. Shiri-Karimvandi, S. Hossein Vahedinejad, M. Doostkamian, E. Monroy and A. Asadolahi. "Investigation of vorticity during prevalent winter precipitation in Iran", Advances in Meteorology, vol. 2018, pp. 1-13, 2018.
[26] D. Schertzer and S. Lovejoy, "Space-time complexity and multifractal predictability", Physica A, vol. 338, pp. 173-186, 2004.
[27] E. Waymire and V. Gupta, "The mathematical Structure of Rainfall Representations 1. A Review of the Stochastic Rainfall Models", Water Resources Research, vol. 17(5), pp. 1261-1272, 1981.
[28] E. Waymire and V. Gupta, "The mathematical Structure of Rainfall Representations 2. A Review of the Theory of Point Processes", Water Resources Research, vol. 17(5), pp. 1273-1285, 1981.
[29] E. Waymire and V. Gupta, "The mathematical Structure of Rainfall Representations 3. Some Applications of the Point Process Theory to Rainfall Model", Water Resources Research, vol. 17(5), pp. 1287-1294, 1981.
[30] D. Veneziano, P. Furcolo and V. Iacobellis, "Imperfect scaling of time and space-time rainfall", Journal of Hydrology, vol. 322, pp. 105-119, 2006.